Bi-Chromatic Minimum Spanning Trees

Magdalene Grantson Henk Meijer David Rappaport*

Abstract

Let G be a set of disjoint bi-chromatic straight line segments and H be a set of red and blue points in the plane, no three points are collinear. We give tight upper bounds on the maximum degree of a node in the color conforming minimum weight spanning tree (MST) formed by G and H. We also consider bounds on the total length of the edges of 1) the planar MST and the unrestricted MST, 2) the greedy planar spanning tree and the unrestricted MST, 3) the greedy planar spanning tree and the planar MST.

1 Introduction

Let G be a set of disjoint bi-chromatic straight line segments, and H a set of red and blue points in the plane, no three points are collinear. We obtain a spanning tree T of G (resp. H) by finding a set of $|G|-1$ (resp. $|H|-1$) edges, which connect vertices of different colors ("color conforming") and form an acyclic connected component. (For the spanning tree of G, input edges and non-input edges alternate, non-input edges may intersect, but are not allowed to intersect input edges.) If T must not contain intersections we call it a *planar spanning tree*,[1] otherwise we simply call T an (unrestricted) *spanning tree*.

A color conforming minimum weight spanning tree (MST) of G or H is a spanning tree of minimum total length of the added edges. Variants of this problem include finding the MST of a set of (mono-chromatic) points or line segments in the plane [2, 3, 1]. For these variants a greedy algorithm like Kruskal's [8] is known to yield the optimal solution [4, 3]. In [2, 3, 9] it is shown that in these cases the maximum degree of a node in the MST is bounded by five and seven, respectively.

Little is known about the MSTs of G or H. [6] gives a survey on geometry graphs of H. Recently, it was shown that a color conforming spanning tree of G or H is always obtainable [5]. It was also shown by illustration that the MST of a given G may contain intersections if one uses a greedy algorithm like Kruskal's [4]. (Kruskal's algorithm adds edges in increasing length order, and discards edges creating a cycle in the graph built so far [8].) We show by illustration that Kruskal's algorithm may also introduce intersections, when H is given as input. Modifying Kruskal's algorithm to check for such intersections and eliminating them, leads to a greedy algorithm which we will refer to as greedy planar algorithm. See [1, 7] for different flavors on non-crossing spanning trees.

$\sup_H \left[\frac{P_{gp}}{P_{op}}\right] \geq 2$	$\sup_G \left[\frac{L_{gp}}{L_{op}}\right] = \infty$	$\sup_G \left[\frac{L_{gp}-L_s}{L_{op}-L_s}\right] \geq 2$
$\sup_H \left[\frac{P_{gp}}{P_{uo}}\right] \geq 2$	$\sup_G \left[\frac{L_{gp}}{L_{uo}}\right] = \infty$	$\sup_G \left[\frac{L_{gp}-L_s}{L_{uo}-L_s}\right] \geq 2$
$\sup_H \left[\frac{P_{op}}{P_{uo}}\right] \geq 2$	$\sup_G \left[\frac{L_{op}}{L_{uo}}\right] = \infty$	$\sup_G \left[\frac{L_{op}-L_s}{L_{uo}-L_s}\right] \geq \frac{3}{2}$

Table 1: An overview of some of our results.

Definition 1 *Given a set G (resp. H) as defined above, we denote by: L_{s} the total length of the input straight line edges/segments, L_{ou} (resp. P_{ou}) the total length of the edges of the unrestricted MST, L_{op} (resp. P_{op}) the total length of the edges of the MST and L_{gp} (resp. P_{gp}) the total length of the edges of the greedy planar spanning tree.*

Summary of our results:
1) We show a bound for the maximum node degree in a color conforming MST.
2) We show the given bounds in Table 1.

2 Maximum Node Degree

Lemma 1 *The maximum degree of a node in a color conforming MST of a set G of n bi-chromatic disjoint line segments is not upper bounded by a fixed number, but only by the size of the input.*

Proof. Trivially the maximum degree of a node of the MST of a set of bi-chromatic disjoint line segments is bounded by n, because every vertex of one color can be connected to at most n vertices of the other color. This upper bound is tight, because an example achieving this bound is shown in Figure 1. For the set G of line segments in Figure 1, suppose the vertex v is red in color and the vertices $\{a, b, c, d, e, f, w\}$ are all blue in color and are at a distance r_1 from v. We refer to these vertices as lying on an inner circle C_i

*Dept. of Computer Science, Queens University, Kingston, Ontario, Canada {grantson,devor,henk}@cs.queensu.ca

[1] Note that the MST of the set of points or line segments in the plane can not contain intersections [3, 5]. For bi-chromatic straight line segments, however, intersections may occur.

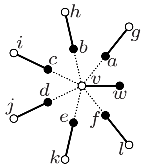

Figure 1: Any spanning tree where the vertex v has a smaller degree than the number of line segments does not have minimum total length.

of radius r_1 with v as its center. Suppose the vertices $\{g, h, i, j, k, l\}$ are red in color and are each at a distance $r_2 = 2r_1$ from v. We refer to these vertices as lying on an outer circle C_o of radius $r_2 = 2r_1$ s.t. v is its center. We make the following observations:

1) There are exactly $n-1$ color conforming edges, each of which has length r_1, namely $\{(v,a),(v,b),(v,c),(v,d),(v,e),(v,f)\}$. Selecting these $n-1$ edges gives a spanning tree.

2) All red vertices visible from an arbitrary blue vertex on the inner circle are $r_1+\epsilon$ away from this vertex, with the exception of v, which is r_1 away.

3) All blue vertices visible from an arbitrary red vertex on the outer circle are $r_1+\epsilon$ away from this vertex.

4) All vertices on the inner circle (outer circle) have the same color, hence it is impossible for an edge to be placed between any pair, since the color conforming characteristics would be violated if we did so.

From the above listed observations, selecting any subset of $n-1$ edges other than the edges from v to the vertices on the inner circle must include at least one edge with a length greater than r_1. Therefore no such subset of edges can be a minimum weight spanning tree. Consequently, the spanning tree formed by the above edges in Figure 1 is of minimum length. □

Lemma 2 *The maximum degree of a node in a color conforming MST of a set of n red and n blue points is not upper bounded by a fixed number, but only by n.*

Proof. Let the set H contain all the vertices in Figure 1, but none of the line segments. Using similar arguments as in the proof of Lemma 1, we observe that there are exactly $2n-1$ color conforming edges having length r_1. Selecting these $2n-1$ edges gives a MST, since all other edges apart from those has length $r_1+\epsilon$, $\epsilon > 0$. □

3 Bounds on Variants of the MST Problem

In [5] it was shown by illustration that the MST of bichromatic line segments may introduce intersections using a greedy algorithm like Kruskal's [4]. We show by illustration in Figure 2 (second diagram) that such intersections may occur when given a set of red and blue points in the plane. Such intersections can however be avoided by modifying, for example, Kruskal's greedy algorithm, so that at each step we rather add the non-crossing edge with the least weight, which does not introduce cycles. Examples in this section clearly show that the modified algorithm may not give the optimal solution. The question then is whether there is a bound on the greedy planar solution in the worst case. We investigate such questions and more. Our results are collected in Table 1.

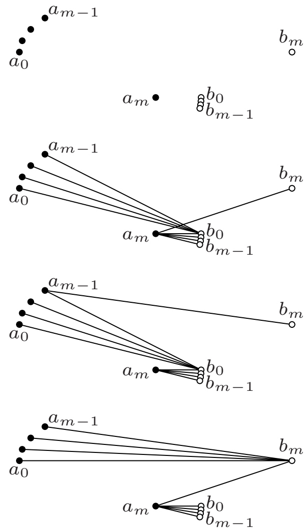

Figure 2: A set of bi-chromatic points for which the greedy spanning tree is by a factor of 2 worse than the planar MST and the unrestricted MST (top: set of points; second: unrestricted MST; third: planar MST; bottom: greedy planar spanning tree).

Observation 1 *If the solution may contain edge intersections, then a greedy approach (e.g. Kruskal's algorithm) always yields the optimal solution.*

Given is a graph $G = (V, E)$ (resp. $H = (V, E)$), such that V is the set of bi-chromatic line segment endpoints (resp. red and blue points) and E is the set of lines of sight between red and blue points. The proof that Kruskal's algorithm finds a minimum spanning tree of G (resp. H) follows from the proof [4] that Kruskal's algorithm finds a minimum spanning tree of any weighted, connected, undirected graph.

3.1 Input is a Set of Red and Blue Points

Theorem 3 *Let H be any set of red and blue points in the plane. Then $\sup_H\left[\frac{P_{gp}}{P_{op}}\right] \geq 2$ and $\sup_H\left[\frac{P_{gp}}{P_{ou}}\right] \geq 2$.*

Proof. Choose $m \in \mathbb{N}$, $m \geq 1$, and let $\phi = \arctan\frac{1}{m^2}$. Then define the points (see Figure 2):

$$\begin{array}{ll} \text{red} & \text{blue} \\ a_m = (0,0), & b_m = (m^2, 1), \\ a_k = \left(\frac{k^2}{m^2} - m^2, 1 + \frac{k}{m}\right), & b_k = \left(\cos(\frac{k\phi}{m}), -\sin(\frac{k\phi}{m})\right). \end{array}$$

$\forall k; 0 \leq k < m$. To show the ratios, let $m \to \infty$. Then

$$\lim_{m\to\infty} \frac{P_{gp}}{P_{op}} \geq \lim_{m\to\infty} \frac{6m^2+m}{3m^2+6m} = 2 \quad \text{and}$$

$$\lim_{m\to\infty} \frac{P_{gp}}{P_{ou}} \geq \lim_{m\to\infty} \frac{6m^2+m}{3m^2+4m} = 2. \qquad \Box$$

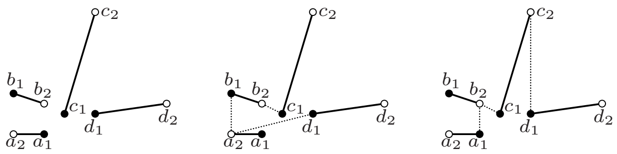

Figure 3: Left: set of bi-chromatic line segments; middle: planar MST; right: greedy planar spanning tree.

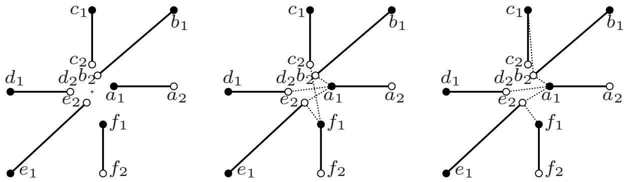

Figure 4: Left: set of bi-chromatic line segments; middle: unrestricted MST; right: planar MST.

Theorem 4 *Let H be any set of red and blue points in the plane. Then* $\sup_H \left[\frac{P_{op}}{P_{ou}}\right] \geq \frac{3}{2}$.

Proof. We consider the set of points

$$\begin{array}{lllllll} & \text{red} & & & \text{blue} & & \\ a_{11} & = & (0,0), & & a_{12} & = & (\epsilon,0), \\ b_{11} & = & (-1,\epsilon), & & b_{12} & = & (1,\epsilon). \end{array}$$

Then

$$\lim_{\epsilon\to 0}\frac{P_{op}}{P_{ou}} = \lim_{\epsilon\to 0}\frac{\epsilon+\sqrt{1+\epsilon^2}+2}{\epsilon+\sqrt{1+\epsilon^2}+\sqrt{\epsilon^2+(1+\epsilon)^2}} = \frac{3}{2}. \quad \square$$

3.2 Input is a Set of Bi-chromatic Line Edges

Theorem 5 *Let G be any set of bi-chromatic straight line segments in the plane. Then* $\sup_G \left[\frac{L_{gp}-L_s}{L_{op}-L_s}\right] = \infty$ *and* $\sup_G \left[\frac{L_{gp}-L_s}{L_{ou}-L_s}\right] = \infty$.

Proof. We consider the set of bi-chromatic line segments in Figure 3 left: $a_1 = (-2,-2)$, $a_2 = (-5,-2)$, $b_1 = (-5,2)$, $b_2 = (-2,1)$, $c_1 = (0,0)$, $c_2 = (3,x)$, $d_1 = (3,0)$, and $d_2 = (x,1)$ with $x \geq 10$. The planar MST consists of the edges (a_2,b_1), (b_2,c_1), and (a_2,d_1) and has a total length of ≈ 13.98, independent of the value of x (see Figure 3 middle).

The modified Kruskal algorithm, however, considers the edge (a_1,b_2) before the edge (a_2,d_1). As a consequence it finds the spanning tree shown in Figure 3 right, which consists of the edges (a_1,b_2), (c_1,b_2), and (d_1,c_2) and thus has a total length of $3+\sqrt{3}+x > 14.73$ for $x \geq 10$. Therefore we have $\frac{L_{gp}-L_s}{L_{op}-L_s} = \frac{3+\sqrt{3}+x}{4+\sqrt{3}+\sqrt{68}}$, which goes to infinity as x goes to infinity. Trivially it follows that $\sup_G \left[\frac{L_{gp}}{L_{ou}}\right] = \infty$, since $L_{op} \geq L_{ou}$. $\square$

Theorem 6 *Let G be any set of bi-chromatic straight line segments in the plane. Then* $\sup_G \left[\frac{L_{op}-L_s}{L_{ou}-L_s}\right] = \infty$.

Proof. We consider the set of bi-chromatic line segments in Figure 4 left. It is $a_1 = (4,1)$, $a_2 = (x,1)$, $b_1 = (x,x)$, $b_2 = (1,3)$, $c_1 = (0,x)$, $c_2 = (0,5)$, $d_1 = (-x,0)$, $d_2 = (-4,0)$, $e_1 = (-x,-x)$, $e_2 = (-1,-2)$, $f_1 = (2,-6)$, $f_2 = (2,-x)$ with $x \geq 15$. The MST consists of the edges (b_2,a_1), (e_2,f_1), (e_2,a_1), (d_2,a_1) and (c_2,f_1) and has a total length of ≈ 33.67, independent of the value of x (see Figure 4 middle). The planar MST, however, consist of the edges (b_2,a_1), (e_2,f_1), (e_2,a_1), (d_2,a_1) and (c_1,b_2), since any other connection leads to an intersection (see Figure 4 right). As a consequence the planar MST has a total length ≥ 34.54 for $x \geq 15$. The ratio of the two total lengths goes to infinity as x goes to infinity. $\square$

Theorem 7 *Let G be any set of bi-chromatic straight line segments in the plane. Then,* $\sup_G \left[\frac{L_{gp}}{L_{ou}}\right] \geq 2$ *and* $\sup_G \left[\frac{L_{op}}{L_{ou}}\right] \geq 2$.

Proof. Choose $m \in \mathbb{N}$, $m \geq 3$, $s \in \mathbb{R}$, $s > 3$, and $r \in [\frac{3}{4},1)$ and let $t(k) = \sqrt{1-(\frac{rk}{m})^2}$. Then define the line segments (see Figure 5):

$$\begin{array}{llll} \text{red} & & \text{blue} & \\ a_{11} = \left(\ \frac{1}{3m}, 0\right), & & a_{12} = (\ \ s+1, -1), & \\ a_{21} = \left(-\frac{1}{3m}, 0\right), & & a_{22} = (-s-1,\ \ 1), & \\ b_{k1} = \left(\ \frac{rk}{m},\ \ t(k)\right), & & b_{k2} = \left(\ \frac{rk}{m}, s+t(k)\right), & \\ c_{k1} = \left(-\frac{rk}{m}, -1-t(k)\right), & & c_{k2} = \left(-\frac{rk}{m},\ \ -t(k)\right). & \end{array}$$

$\forall k; 1 \leq k \leq m$. To show the above ratios, we let $m \to \infty$ and $s \to \infty$. To simplify the double limit $m \to \infty$ and $s \to \infty$, choose $s = m^2$. This yields

$$\lim_{m\to\infty}\frac{L_{gp}}{L_{ou}} = \lim_{m\to\infty}\frac{L_{op}}{L_{ou}} \geq \lim_{m\to\infty}\frac{2m^3}{m^3+13m^2} = 2. \quad \square$$

Theorem 8 *Let G be any set of bi-chromatic straight line segments in the plane. Then* $\sup_G \left[\frac{L_{gp}}{L_{op}}\right] \geq 2$.

Proof. Choose $m \in \mathbb{N}$, $m \geq 3$, $s \in \mathbb{R}$, $s \geq 6$, and let $t(k) = \sqrt{5-(\frac{k}{m})^2}$. Then define the line segments

$$\begin{array}{ll} \text{red} & \text{blue} \\ a_{11} = (\ \ 1, 1), & a_{12} = (1+s, 1), \\ a_{21} = (\ \ 0, 3), & a_{22} = (\ \ -s, 3), \\ b_{11} = (\ \ 2, 0), & b_{12} = (\ \ \ \ 0, 0), \\ b_{21} = (-3, 2), & b_{22} = (\ \ -1, 2), \\ c_{k1} = \left(\frac{k}{m}, t(k)\right), & c_{k2} = \left(\frac{k}{m}, s+t(k)\right), \end{array}$$

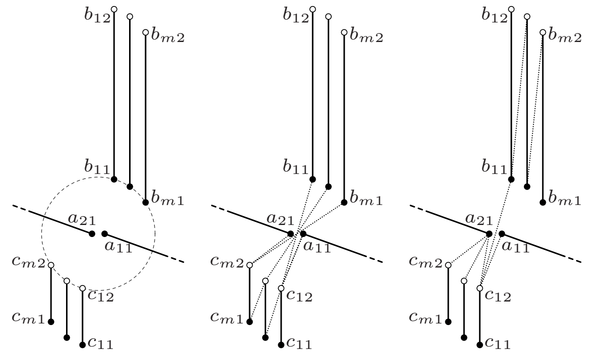

Figure 5: Left: set of bi-chromatic line segments; middle: unrestricted MST; right: planar MST.

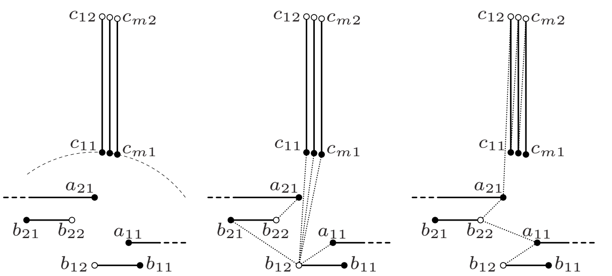

Figure 6: Left: set of bi-chromatic line segments; middle: planar MST; right: greedy planar spanning tree.

$\forall k; 1 \leq k \leq m$. To show the above ratios, we let $m \to \infty$ and $s \to \infty$. To simplify the double limit $m \to \infty$ and $s \to \infty$, choose $s = m^2$. This yields

$$\lim_{m\to\infty} \frac{L_{gp}}{L_{op}} = \lim_{m\to\infty} \frac{L_{gp}}{L_{op}} \geq \lim_{m\to\infty} \frac{2m^3 + 2m^2}{m^3 + 2m^2 + 40} = 2. \qquad \square$$

3.3 Lower Bounds

Trivially the lower bounds of all the ratios in Table 1 is 1, because the numerator in each ratio is never less than the denominator. Moreover, instances, where the numerator in each ratio equal to the denominator, can be constructed.

3.4 Upper Bounds

Lemma 9 *The upper bound of the ratios* $\frac{P_{gp}}{P_{op}}$, $\frac{P_{gp}}{P_{ou}}$, $\frac{P_{op}}{P_{ou}}$, $\frac{L_{gp}}{L_{op}}$, $\frac{L_{gp}}{L_{ou}}$, *and* $\frac{L_{op}}{L_{ou}}$ *is* n.

Proof. Let d denote the length of the diagonal of the box bounding a given set G of bi-chromatic line segments or H of red and blue points in the plane. Trivially, $L_{ou} \geq d$ and $P_{ou} \geq d$. The total length of edges T_G of any tree of G is $\leq (2n-1)d$. Similarly, the total length of edges T_H of any tree of H is $\leq (n-1)d$ (see Figure 7). Hence we have $\frac{P_{gp}}{P_{op}} \leq n$, $\frac{P_{gp}}{P_{ou}} \leq n$, $\frac{P_{op}}{P_{ou}} \leq n$, $\frac{L_{gp}}{L_{op}} \leq n$, $\frac{L_{gp}}{L_{ou}} \leq n$, and $\frac{L_{op}}{L_{ou}} \leq n$. $\square$

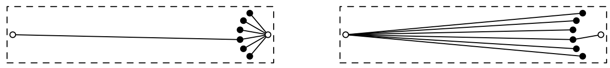

Figure 7: Set of bi-chromatic points for the upper bounds of the ratios.

4 Open Problem

An algorithm to determine a color conforming planar MST of a set of bi-chromatic line segments or red and blue points within a factor k still remains open. Moreover upper bounds other than n on the ratios: $\frac{P_{gp}}{P_{op}}$, $\frac{P_{gp}}{P_{ou}}$, $\frac{P_{op}}{P_{ou}}$, $\frac{L_{gp}}{L_{op}}$, $\frac{L_{gp}}{L_{ou}}$, $\frac{L_{op}}{L_{ou}}$, also remains open.

References

[1] O. Aichholzer, F Aurenhammer and F. Hurtado. Sequences of Spanning Trees and a Fixed Tree Theorem. *Computational Geometry: Theory and Applications* 21(1-2):3–20. Elsevier, Amsterdam, Netherlands 2002.

[2] P. Bose and G. Toussaint. Growing a Tree from its Branches. *Proc. 1st Pacific Conf. on Computer Graphics and Applications (Pacific Graphics '93, Seoul, Korea)*, 91–103. World Scientific Publishing Co., Hackensack, NJ, USA 1993.

[3] P. Bose and G. Toussaint. Growing a Tree from its Branches. *Journal of Algorithms* 19(1):86–103. Elsevier, Amsterdam, Netherlands 1995.

[4] T.H. Cormen, C.E. Leiserson, R.L. Rivest, and C. Stein. Introduction to Algorithms (Second Edition). MIT Press and McGraw-Hill book company, USA 2002.

[5] F. Hurtado, M. Kano, D. Rappaport and C.D. Tòth. Encompassing Colored Crossing-Free Geometric Graphs. *16th Canadian Conference on Computational Geometry.* Montreal, Canada 2004.

[6] A. Kaneko and M. Kano. Discrete Geometry on Red and Blue Points in the Plane — A Survey. Discrete and Computational Geometry, *The Goodman-Pollack Festschrift, With contributions by numerous experts* 551–570. Springer, Heidelberg, Germany 2003.

[7] G. Károlyi, J. Pach, G. Tóth. Ramsey-Type results for geometric graphs. *Discrete and Computational Geometry* 18:247–255. 1997.

[8] J.B. Kruskal. On the Shortest Spanning Subtree of a Graph and the Traveling Salesman Problem. *Proc. of the American Mathematical Society,* 7:48–50. 1956.

[9] C. Monma and S. Suri. Transitions in Geometric Minimum Spanning Tree. *Bell Communications Research Technical Report.* Bell Communications Research, Inc., Livingstone, NJ, USA 1991.