Algebraic Study of the Apollonius Circle of Three Ellipses

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Abstract

We study the external tritangent Apollonius (or Voronoi) circle to three ellipses. This problem arises when one wishes to compute the Apollonius (or Voronoi) diagram of a set of ellipses, but is also of independent interest in enumerative geometry. This paper is restricted to non-intersecting ellipses, but the extension to arbitrary ellipses is possible.

We propose an efficient representation of the distance between a point and an ellipse by considering a parametric circle tangent to an ellipse. The distance of its center to the ellipse is expressed by requiring that their characteristic polynomial have at least one multiple real root. We study the complexity of the tritangent Apollonius circle problem, using the above representation for the distance, as well as sparse (or toric) elimination. We offer the first nontrivial upper bound on the number of tritangent circles, namely 184.

Keywords: Voronoi diagram, ellipse, mixed volume, Euclidean distance, resultant.

1 Introduction

Voronoi diagrams are interesting constructs with numerous applications and have been studied extensively. However, the bulk of the existing work in the plane concerns point-sites or linear sites such as segments and polygons. More recently, some works have extended Apollonius (or Voronoi) diagrams to the case of circles, e.g. [8]. For the latter problem, the implementation of [3] is now part of the CGAL library. Recent works derive (semi-)algebraic conditions for characterizing the relative position of conics in the plane or certain quadrics in space, e.g. [12], [5].

Our ultimate goal is to compute efficiently and exactly the Apollonius (or Voronoi) diagram of arbitrary sets of ellipses in the plane, under the Euclidean metric. We assume that the ellipses are given algebraically, or implicitly. This is clearly a harder problem than the diagram of circles or the visibility map among ellipses, hence the need for higher degree algebraic operations. As a first step, this paper studies the case of *non-intersecting* ellipses, although our methods readily extend to arbitrary inputs.

The algorithms for the Apollonius diagram of ellipses typically use the following 2 main predicates. Further predicates are examined in [4].

(1) given two ellipses and a point outside of both, decide which is the ellipse closest to the point, under the Euclidean metric

(2) given 4 ellipses, decide the relative position of the fourth one with respect to the external tritangent Apollonius circle of the first three

For predicate (1) we consider a circle, centered at the point, with unknown radius, which corresponds to the distance to be compared. A tangency point between the circle and the ellipse exists iff the discriminant of the corresponding pencil's determinant vanishes. Hence we arrive at a method using algebraic numbers of degree 4, which is optimal. Note that we avoid expressing the coordinates of the tangency point.

Let us focus on predicate (2). In the case of 3 disks, the number of tritangent circles is 8 and the corresponding predicate is of algebraic degree 2 [3]. This problem is also known as *The circle of Apollonius*, because it was first addressed by *Apollonius of Perga*, in about 250 BC. While this has been known since antiquity, the generalization to ellipses is yet to be solved efficiently.

Even the number of tritangent circles to 3 ellipses is not known. The problem involves equations of high degree and obtaining an exact solution is nontrivial. [10] attempts to deal with this problem, but exact computation with the proposed method is not completed and the author reverts to numerical methods. No bounds on the complexity are given, nor on the number of tritangent circles.

We apply the method from predicate (1) and recent advances in sparse (or toric) resultants in order to project all common roots to those of a univariate equation. This leads to the first interesting bound on the number of tritangent circles, namely 184. Mixed volume also gives this bound as does a real algebraic geometry argument [11].

The paper is organized as follows. The next section introduces some of the ellipse's properties. In section 3 we describe an efficient representation for the Euclidean distance between a point and an ellipse and we apply this idea to predicate (1). Finally, section 4 deals directly with the external tritangent Apollonius circle and predicate (2).

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2 The geometry of an ellipse

An *ellipse* is the locus of points in the plane the sum of whose distances from the foci is $2\alpha > 0$. The foci lie at distance 2γ. The length of the major and minor axes are $2\alpha, 2\beta$, resp. where $\beta^2 = \alpha^2 - \gamma^2$. Let (x_c, y_c) be its center and u the angle between the major and the x axes. When $x_c = y_c = u = 0$ the ellipse is in *orthogonal* position, otherwise, it is in *generic* position. The ellipse is a *conic* section with equation:

$$E(x, y) := ax^2 + 2bxy + cy^2 + 2dx + 2ey + f, \quad (1)$$

where $J_2 > 0 \neq ac$ and J_2 is defined below. The ellipse's parameters (a, b, c, d, e, f) are related to its center, rotation, axes and focal distance, but we omit the corresponding equations. The following quantities are *invariants* under rotation and translation:

$$J_1 = a+c = \alpha^2+\beta^2 > 0, \quad J_2 = \begin{vmatrix} a & b \\ b & c \end{vmatrix} = \alpha^2\beta^2 > 0,$$

$$J_3 = \begin{vmatrix} a & b & d \\ b & c & e \\ d & e & f \end{vmatrix} = -J_2^2 < 0.$$

The following quantity is invariant under rotation; its expression uses the lemma below.

$$J_4 = (a + c)f - d^2 - e^2 = J_2(x_c^2 + y_c^2 - J_1).$$

Let L_y, L_x be the lines connecting the leftmost and rightmost points, and the highest with the lowest point, respectively.

Lemma 1 *Consider an ellipse of the form (1). Its center is rational and coincides with the intersection of* L_x, L_y*, where* $x_c = (be - dc)/J_2$, $y_c = (bd - ae)/J_2$.

Given a point V outside an ellipse, how many normals are there to the ellipse? Let us count normal *segments*, defined as the segment of a line normal to the ellipse at some point Q; the segment's endpoints are Q, V. The boundary of the regions where the number of normals changes is the *evolute*, which is a stretched astroid (see figure 1). For an ellipse in orthogonal position, each point (x, y) on the evolute satisfies: $(\alpha x)^{\frac{2}{3}} + (\beta y)^{\frac{2}{3}} = \gamma^{\frac{4}{3}}$.

Proposition 2 *There are 4, 3 or 2 normals of a point to an ellipse, depending on whether the point lies inside the evolute, lies on the evolute but not at a cusp or, respectively, the point is a cusp or outside the evolute.*

This yields a lower bound on the algebraic complexity of computing the distance of an external point to the ellipse, since any condition on the unknown distance has degree ≥ 4.

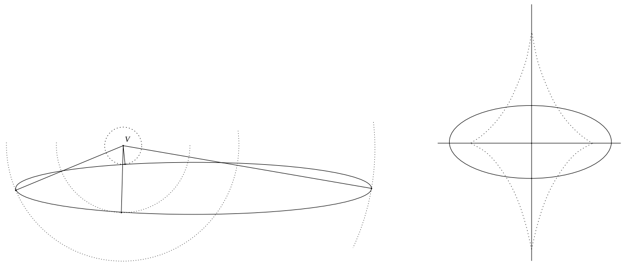

Figure 1: Left: an example of a point with 4 normals. Right: the evolute of an ellipse.

3 Distance between point and ellipse

Consider an ellipse E and a point $V = (v_1, v_2)$ outside E. Let C be a circle centered at V with radius equal to $\sqrt{s}$, for a real $s > 0$. We shall express the *Euclidean distance* $\delta(V, E)$ between V and E by the *smallest* positive value of $\sqrt{s}$ for which C is tangent to E. In comparing distances, it is sufficient to consider squared distance s.

It would be possible to find all tangency points by solving the system:

$$E(x, y) = \det[\nabla E(x, y), (x, y) - V] = 0, \quad (2)$$

and then choosing the appropriate solution, where the 2nd equation constraints the vector $(x, y) - V$ to be normal to E at (x, y). Consider system (2) with an additional equation: $\|(x, y) - V\|^2 = s$. The resultant of the 3 polynomials with respect to x, y is precisely polynomial $\Delta(v_1, v_2, s)$ to be defined below by an alternative manner. It is the algebraic representation of the offset curve to E at distance s.

Our goal is to avoid explicit computation of the tangency points by requiring that system $E = C = 0$ have a multiple root. To arrive at a simple polynomial we apply the theory of characteristic polynomials and pencils [9, 12]. Let us express a conic as $[x, y, 1]M[x, y, 1]^T$, for an appropriate matrix M. Then E, C correspond to

$$A = \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & -v_1 \\ 0 & 1 & -v_2 \\ -v_1 & -v_2 & v_1^2 + v_2^2 - s \end{bmatrix}$$

The *pencil* of E and C is $\lambda A + B$, and their characteristic polynomial is

$$\begin{aligned} \phi(\lambda) &= |\lambda A + B| = J_2^2\lambda^3 + c_2\lambda^2 + c_1\lambda + s, \\ c_2(s) &= J_2 s - T(v_1, v_2), \\ c_1(s) &= J_1 s - E(v_1, v_2), \\ T(v_1, v_2) &= J_2[(v_1 - x_c)^2 + (v_2 - y_c)^2 - J_1]. \end{aligned}$$

The discriminant $\Delta(s)$ of $\phi(\lambda)$ is of degree 4:

$$\begin{aligned}\Delta(s) = J_2^2(J_1^2 - 4J_2)\, s^4 + \\ 2J_2(9J_1J_2^2 - J_1^2T + 6J_2T - 2J_1^3J_2 - J_1J_2E)\, s^3 + \\ +(-18J_2^3E + 4J_1J_2ET - 27J_2^4 + J_1^2T^2 - \\ -18J_1J_2^2T + J_2^2E^2 + 12J_1^2J_2^2E - 12J_2T^2)\, s^2 + \\ 2(2T^3 - J_1ET^2 - 6J_1J_2^2E^2 + 9J_2^2ET - J_2E^2T)\, s \\ +E^2(T^2 + 4J_2^2E).\end{aligned}$$

The relative position of a circle and an ellipse falls into one of 9 cases, related to the multiplicity and signs of the real roots of $\phi(\lambda)$ [12, thm.8]. When $\phi(\lambda)$ has at least one multiple root, the ellipse and the circle have at least one tangency point. Note that $\phi(\lambda)$ has at least one negative root because the product of roots equals $-s < 0$.

By picking the smallest positive root of $\Delta(s) = 0$, we assure that $\phi(\lambda)$ has at least one root with multiplicity greater than one. Assume that the circle centered at V grows until it touches E (and then it might continue to grow until it fully contains E). Since V is outside E, the smallest positive root of $\Delta(s)$ corresponds to $\delta(V, E)$.

Proposition 3 *Given an ellipse E and a point V outside E, $\delta(V, E)$ is the square-root of the smallest positive zero of $\Delta(s)$; the latter is a univariate polynomial of degree 4. The degree of the coefficients of $\Delta(s)$ is 6, 8, 10, 12, and 14, in order of decreasing power in s, in v_1, v_2 and the parameters of E. In case (v_1, v_2) is the center of another ellipse E' the degree of the coefficients of $\Delta(s)$ is exactly 22 in the parameters of E, E'.*

Corollary 4 *Given ellipses E_1, E_2 and point V outside both of them, we can decide which ellipse is closest to V by comparing two algebraic numbers of degree 4. The previous section implies that this degree is optimal.*

4 External tritangent circle

Given 3 ellipses in the form of equation (1) we want to find an external tritangent circle, as shown in fig. 2. Eventually, we are interested in deciding on the relative position of a fourth ellipse and the circle. An important open question is: what is the maximum number of real tritangent circles to 3 ellipses? Given the discussion in section 2, we expect that there are at least $4^3 = 64$ such circles.

Let $\sqrt{s}$ be the radius of the tritangent circle and (v_1, v_2) its center. Using the discriminant as above for each of the 3 ellipses, we get

$$\Delta_1(v_1, v_2, s) = \Delta_2(v_1, v_2, s) = \Delta_3(v_1, v_2, s) = 0. \quad (3)$$

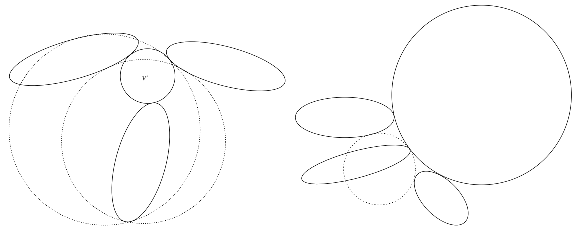

Figure 2: Tritangent circles to 3 ellipses; only one is externally tangent

Among the solutions of this system, the external tritangent circle of interest may or may *not* have the smallest radius; cf. the respective cases in figure 2.

We apply sparse (or toric) elimination theory, using the properties of the resultant and the mixed volume. Given a system of $n+1$ polynomials f_i in n variables, with coefficients c_{ij}, the *resultant* of these polynomials is a new polynomial $R \in \mathbb{Z}[c_{ij}]$ such that when c_{ij} are specialized, $R = 0 \iff \exists a : \forall i \; f_i(a) = 0$. If the roots a lie in a projective (resp. toric) variety, then we refer to the projective (resp. toric) resultant. Given n polynomials in n variables, the *mixed volume* of this polynomial system is a function of the support (Newton polytope) of each polynomial. The mixed volume provides an upper bound on the roots of the system in $(\mathbb{C}^*)^n$. For more information see [2].

Each Δ_i is of total degree 8 in v_1, v_2, s and 4 in s. The mixed volume of system (3) is 256, which is too high. It is known that this bound may not be tight, as it may count complex roots and roots at "infinity".

Indeed, it is possible to reduce the mixed volume of the above system, in order to obtain a better upper bound. We set

$$q := v_1^2 + v_2^2 - s \quad (4)$$

In this case, the matrix B defined in the previous section, contains only linear terms with respect to v_1, v_2, q. The discriminant of the characteristic polynomial is of total degree 6 in v_1, v_2, q and 4 in q; the coefficients of $1, q, q^2, q^3, q^4$ are polynomials in v_1, v_2 of degree 6,5,4,2,1 respectively. The corresponding system has mixed volume 184. Note that solving for v_1, v_2, s requires the use of equation (4). The mixed volume of the system of Δ_i with this additional equation, with respect to v_1, v_2, s, q, is still 184.

Recent advances in matrix formulae for the resultant allow us to compute the resultant of certain systems of 3 bivariate polynomials as a single determinant. One class of such systems are those with identical supports [7]. The corresponding matrix is of hybrid type, i.e., it contains blocks of Sylvester and of Bézout type. The matrix construction has been implemented in Maple by A. Khetan. Our system (3) falls in this class, considered with variables v_1, v_2, after hiding q in the coefficients. Its resultant is a polynomial in q and equals, generically, the determinant of a 58×58 matrix. We denote this matrix by K.

To get an idea of the quantities involved, we have studied a specific example of three ellipses, in random position as in the left-hand side figure 2. The input parameters are signed 10-bit integers. The elements of K are either 0 or polynomials in q of degree 0–10. The computation of the determinant of K is done by interpolation. The determinant of K is a polynomial in q, which we denote by $d(q)$. By substituting different values of q into K we eliminate all indeterminates making the computation of the determinant a trivial task. By making 200 such replacements (in fact, 185 suffice) we obtain 200 pairs of $\langle q, d(q)\rangle$. It turns out that there is a unique interpolating polynomial of degree 184 in q through these values which is exactly the resultant of our example. Hence, in this example the number of complex solutions matches the upper bound given by the mixed volume.

The coefficients of this resultant are, on average, 1385-digit (4603-bit) integers. We 've not yet managed to solve this resultant efficiently and exactly. However, as a preliminary approach we have applied the Aberth method (implemented in [1]) to solve the polynomial numerically. This algorithm yielded 8 real roots in less than a second.

According to [11], there are 184 complex circles in the worst case that are tangent to 3 given conics in the plane. The idea is to consider a manifold (space of complete conics) whose cohomology ring ([6]) has two generators:

$$p \;:=\; \text{a conic contains a fixed, but general point}$$
$$l \;:=\; \text{a conic is tangent to a fixed, but general line}$$

In this ring, conjunction of conditions is multiplication, and every degree-5 monomial is associated to an integer. For example $p^4 l \mapsto 2$ meaning that there are 2 conics through 4 points and tangent to a line. The condition of tangency to a conic is $2(p + l)$ and a circle contains the two circulars points at infinity $(x_1 : x_2 : x_0) = (i : \pm 1 : 0)$. Thus the expression $p^2[2(p + l)]^3$ maps to 184 which means that there are at most 184 complex circles tangent to 3 conics in the plane. An open question is how many of these circles can be real.

The above computation of the mixed volume, the resultant, and the upper bound give strong indication that the system (3) modified with (4) is optimal. Methods on how to solve system (3) efficiently will be addressed in a future work. In such an approach, one might consider semi-algebraic constraints such as those in [4], in order to prune cases in some subdivision-based algorithm.

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References

[1] D.A. Bini and G. Fiorentino. Design, analysis, and implementation of a multiprecision polynomial rootfinder. *Numerical Algorithms*, 23:127–173, 2000.

[2] D. Cox, J. Little, and D. O'Shea. *Using Algebraic Geometry*. Number 185 in Graduate Texts in Mathematics. Springer-Verlag, New York, 1998.

[3] I.Z. Emiris and M.I. Karavelas. The predicates of the Apollonius diagram: algorithmic analysis and implementation. *Comp. Geom.: Theory & Appl., Spec. Issue*, 2005. To appear.

[4] I.Z. Emiris and G.M. Tzoumas. Distance predicates among ellipses. Manuscript. www.di.uoa.gr/~grad0491/msc/DistPredEll.pdf, 2004.

[5] F. Etayo, L. Gonzalez-Vega, and N. del Rio. A new approach for characterising the relative position of two ellipses depending on one or several parameters. 2004. Submitted.

[6] J. Harris. *Algebraic Geometry (A First Course)*. Springer-Verlag, Berlin, Germany, 1992.

[7] A. Khetan. The resultant of an unmixed bivariate system. *J. Symb. Comput.*, 36:425–442, 2003.

[8] D.-S. Kim, D. Kim, and K. Sugihara. Voronoi diagram of a circle set from Voronoi diagram of a point set: II. Geometry. *CAGD*, 18:563–585, 2001.

[9] H. Levy. *Projective and Related Geometries*. McMillan Co., 1967.

[10] R. Lewis and S. Bridget. Polynomial equations arising from apollonius problems in biochemistry and pharmacology. In *Proceedings of Applications of Computer Algebra*, 2001.

[11] F. Sottile. Personal communication. 2004.

[12] W. Wang, J. Wang, and M. Kim. An algebraic condition for the separation of two ellipsoids. *Comp. Aided Geom. Design*, 18:531–539, 2001.