On Properties of Higher-Order Delaunay Graphs with Applications*

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Abstract

In this work we study the order-k Delaunay graph, which is formed by edges pq having a circle through p and q and containing no more than k sites. We study the combinatorial structure of the set of triangulations that can be constructed with edges of this graph and show that it is connected under the flip operation if $k \leq 1$ and for every k if points are in convex position. We also study the hamiltonicity of the order-k Delaunay graph and give an application to a coloring problem.

1 Introduction

The Delaunay graph is an ubiquitous structure in the field of Computational Geometry. It is well known that this graph is a triangulation when the points are in general position and that it can be easily completed to a triangulation in the presence of degenerate configurations. An encyclopedic treatment of this structure can be found in the book by Okabe et al. [7].

The edges of a Delaunay triangulation of a planar point set P have a simple geometric definition (i.e. its proximity measure). Two points $p, q \in P$ form a Delaunay edge provided that there exists a circle with p and q on its boundary with no points of $P \setminus \{p, q\}$ in its interior.

This condition can be generalized in a natural way by relaxing the requirement that the circle needs to be empty. In this way, we say that $p, q \in P$ form an edge of the *order-k Delaunay graph* provided that there exists a circle with p and q on its boundary with at most k points of the set $P \setminus \{p, q\}$ inside the circle. Note that the order-0 Delaunay graph is the standard one.

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In [3] the authors don't focus on the order-k Delaunay graph yet its edges are defined and called order-k Delaunay edges; then they deal with the problem of computing the set of order-k Delaunay edges which can be completed to a triangulation such that all the triangles have order at most k, where the order of a triangle is defined as the number of points contained inside its circumscribing circle. For the constrained Delaunay triangulation, related problems are considered in [4].

It may be surprising that similar questions have been considered some years ago for graphs related to the Delaunay graph: In [8], properties of the order-k Gabriel Graph (GG) are investigated and an algorithm for its construction is proposed, while in [1] it is shown that the order-20 Relative Neighborhood Graph (RNG) is Hamiltonian.

In this paper, we concentrate mainly on the study of some graph theoretic properties of the order-k Delaunay graph as well on some applications arising from these properties.

2 Order-k Delaunay graph

Throughout this paper, unless explicitly stated otherwise, P will be a set of points in the plane in general position – no three points are collinear and no four are on a circle.

Definition 1 *Given two points $p, q \in P$, the* order *of pq is the smallest integer k such that there exists a circle through p and q containing in its interior k points of P. The order-k Delaunay graph of P, denoted $k - DG(P)$, is formed by the edges with order at most k.*

We start by giving an upper bound on the number of edges of the order-k Delaunay graph which can be derived taking into account its relation with higher order Voronoi diagrams [7].

Theorem 1 *Let P be a set of points in general position and let $|k - DG(P)|$ be the number of edges of the order-k Delaunay graph. Then*

$$|k - DG(P)| \leq 3(k+1)n - 3(k+1)(k+2)$$

If P is in convex position, then

$$|k - DG(P)| \leq 2(k+1)n - \frac{3}{2}(k+1)(k+2)$$

Proof. Let b_{pq} be the bisector of points p and q and let $V_k(P)$ be the order-k Voronoi diagram of P. Clearly, if the order of pq is k, an edge of b_{pq} appears for the first time in $V_{k+1}(P)$. In [2], it is shown that the total number of connected components that appear in the set of lines $\{b_{pq} \mid p, q \in P\}$ when all Voronoi diagrams up to order k are put together is

$$\lambda_k = 3kn - \frac{3}{2}\, k(k+1) - \sum_{j=1}^{k} e_j(P),$$

where $e_j(P)$ is the number of j-sets of P. If P is in convex position, then $\sum_{j=1}^{k} e_j(S) = kn$, while for arbitrary P is known that

$$\sum_{j=1}^{k} e_j(S) \geq 3\binom{k+1}{2}$$

(see [2],[6]). Therefore, the result follows from the fact that $|k - DG(P)| \leq \lambda_{k+1}$. ☐

3 Flip-graph of order-k triangulations

In this section we study the structure of the set of triangulations that can be constructed using edges of the order-k Delaunay graph. We say that a triangulation T has order k if all its edges have order at most k and there is some edge with order exactly k. We recall that if a triangulation T_1 has two triangles pqr and pqs in convex position, we can get another triangulation T_2 by deleting the edge pq and adding the edge rs. This operation is called a *flip*. In this situation, we say that the edge pq is locally Delaunay if the circle passing through p, q and r does not contain point s.

Definition 2 *The flip-graph of triangulations with order at most k, denoted by $TG_k(P)$, is defined in the following way:*

1. *the vertices are the triangulations of P with order at most k,*
2. *two triangulations T_1 and T_2 are connected with an edge in $TG_k(P)$ if they differ in a flip.*

If $k = 0$, $TG_0(P)$ is a single vertex (the Delaunay triangulation) and thus connected. In the following theorem we answer the question of the connectedness of these graphs.

Theorem 2

a) *$TG_1(P)$ is connected.*

b) *$TG_k(P)$ can be disconnected if $k \geq 2$.*

c) *If P is in convex position, then $TG_k(P)$ is connected for every $k \geq 0$.*

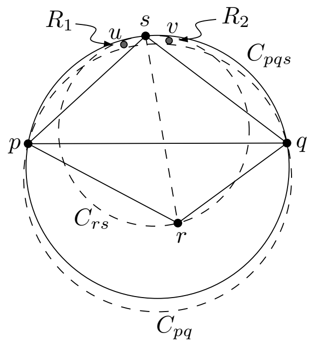

Figure 1: Illustration for the proof of Theorem 2

Proof. Let T be a triangulation with order one and let DT be the Delaunay triangulation of P. We are going to show that if $T \neq DT$ there exists an edge of T which is not locally Delaunay and can be flipped to an edge with order at most one.

Let pq be an edge which is not locally Delaunay (then, it has order one) and let rs be the edge that we get when pq is flipped (see Figure 1). If rs has order at most one then we have done, so assume that rs has order at least two. Because pq is not locally Delaunay and has order one, there exists a circle C_{pq} passing through p and q and containing a single point, which is necessarily either r or s. In the following, we assume that C_{pq} contains r and, therefore, the edges pr and qr are Delaunay edges.

Let C_{pqs} be the circle passing through p, q and s and C_{rs} the circle tangent to C_{pqs} at s and passing through r. Let R_1 and R_2 be the regions inside C_{rs} and outside both of the circle C_{pq} and the quadrilateral $prqs$. It is easy to see that each of the regions contains exactly one point, as illustrated in Figure 1).

Let us denote by u and v, respectively, the points inside the regions R_1 and R_2, and by C_{prs} the circle through p, r and s. The circle C_{prs} contains at least two points and no point different from u and v can be inside it. This shows that the edge pv has order at most one. In an analogous way, it can be seen that the edge qu has order at most one. If the edge ps is not locally Delaunay then we have finished because the triangle psu is in T and we can flip the edge ps to qu so we can assume that ps is locally Delaunay.

If triangle psu is not in T, then we can consider the set of triangles C intersected by segment qu and show that if $p'q's'$ and $p'us'$ are adjacent triangles in C then the edge $p's'$ is not locally Delaunay while the edge $q'u$ has order zero and this concludes the proof of part a).

In Figure 2 we show an example of a triangulation with order two such that every possible flip increases its order to three. Therefore, $TG_2(P)$ is not connected.

The proof of part c) is omitted in this extended abstract due to space limitations. ☐

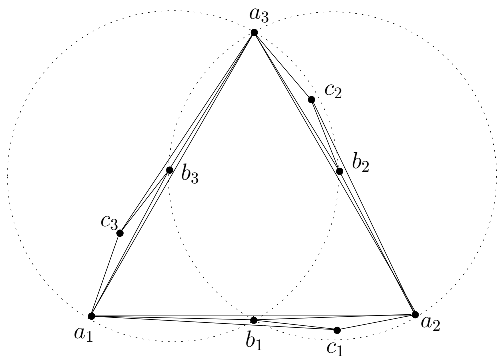

Figure 2: An isolated triangulation in $TG_2(P)$

4 Hamiltonicity of Order-k Delaunay Graph

In this section, we show that the order-15 Gabriel Graph (GG) contains a Hamiltonian cycle. Note that 15-GG is a subgraph of the 15-DG. The key idea behind the proof is the following. Given a particular Hamiltonian cycle h through a set of n points, define the distance sequence, $ds(h) = \delta_1, \ldots, \delta_n$ to be the sequence of edge lengths in the cycle sorted from longest to shortest edge. Given any two Hamiltonian cycles x and y, we can compare lexicographically their edge length sequences. In the following theorem we prove that a cycle associated with an edge length sequence which is minimum with that order has the property that every edge belongs to 15-GG.

Theorem 3 *Given a set P of n points in the plane in general position, the graph 15-GG contains a Hamiltonian cycle (and hence 15-DG too).*

Proof. Let H be the set of all Hamiltonian cycles through the points of P. Let $m = a_0, a_1, \ldots, a_{n-1}$ be a cycle in H with minimal distance sequence. We will show that all of the edges of m are in 15-GG. We proceed by contradiction.

Suppose that there are some edges in m that are not in 15-GG. Let $e = [a_i a_{i+1}]$ be the longest edge that is not in 15-GG (all index manipulation is modulo n). Let B be the circle with a_i and a_{i+1} as diameter.

Claim 1: No edge of m can be completely inside B. Suppose there was an edge $f = [a_j, a_{j+1}]$ inside B. By deleting e and f from m and adding either $[a_i, a_j], [a_{i+1}, a_{j+1}]$ or $[a_i, a_{j+1}], [a_{i+1}, a_j]$, we construct a new cycle m' whose distance sequence is strictly smaller than that of m since $d(a_i, a_{i+1}) > max\{d(a_i, a_j), d(a_{i+1}, a_{j+1}), d(a_i, a_{j+1}), d(a_{i+1}, a_j)\}$. But this is a contradiction since m is a minimal distance sequence.

Therefore, we may assume that no edge of m lies completely inside B. Since e is not 15-GG there must be at least $w \geq 16$ points of P in B. Let $U = u_1, u_2, \ldots, u_w$ represent these points indexed in the order we would encounter them on the cycle starting from a_i. Let $S = s_1, s_2, \ldots, s_w$ and $T = t_1, t_2, \ldots, t_w$ represent the vertices where s_i is the vertex preceding u_i on the cycle and t_i is the vertex succeeding u_i on the cycle.

Let D be the circle centered at a_{i+1} with radius $2r$.

Claim 2: No point of T can be inside D. Suppose $t_j \in T$ is in D, then $d(t_j, a_{i+1}) < 2r$. Construct a new cycle m' by removing the edges $[u_j, t_j], [a_i, a_{i+1}]$ and adding the edges $[a_{i+1}, t_j], [a_i, u_j]$. Since the two edges added have length strictly less than $2r$, $ds(m') < ds(m)$ which is a contradiction.

Let c be the midpoint of the edge $[a_i, a_{i+1}]$. Let C be the circle centered at c with radius $2r$ and

Claim 3: There are at most 4 points of T in C. Suppose that there are 5 points of T in C. Note that the 5 points are in $C \cap \overline{D}$ by the previous claim. However, this means that there must be two points t_j, t_k such that $\angle(t_j, c, t_k) < \pi/3$. But this implies that $|\overline{t_j t_k}| < 2r$.

Since $|T| \geq 15$, there are at least 11 points of T outside C. Decompose the plane into 10 cones of angle $\pi/5$ centered at c. By the pigeon-hole principle, there must be one cone with at least 2 points, t_j and t_k. We note that $d(t_j, t_k)$ is either less than $2r$ or less than $\max d(c, t_j) - r, d(c, t_k) - r$ (a proof of this fact can be found in the technical report). Construct a new cycle m' from m by first deleting $[t_j, u_j], [t_k, u_k], [a_i, a_{i+1}]$. This results in three paths. One of the paths must contain both a_i and either t_j or t_k. WLOG, suppose that a_i and t_j are on the same path. Add the edges $[a_i, u_k], [a_{i+1}, u_j], [t_j, t_k]$. The resulting cycle m' has a strictly smaller distance sequence since $\max [t_j, u_j], [t_k, u_k], [a_i, a_{i+1}] > \max [a_i, u_k], [a_{i+1}, u_j], [t_j, t_k]$.

□

5 Coloring with Applications

Given a set of n points in the plane, Har-Peled and Smorodinsky [5] showed how to assign one of m colors to each of the n points such that every circle C containing more than one point has at least one point in C with a unique color. Such a coloring is called a *conflict-free* coloring (CF-coloring for short). The Delaunay graph is used both in the coloring algorithm and to show that m is $O(\log n)$. This type of coloring finds application in the assignment of frequencies in a cellular network.

In this section, we generalize the result in [5]. We show that with $O(\log n / \log(8ck/(8ck - 1))$ colors, a set of n points in the plane can be colored so that every circle containing at least k points contains at least k points with unique color (where the maximum number of edges in $(k-1)$-DG is ckn for some constant c). We call such a coloring a k-conflict-free coloring. In the context of cellular networks, this can be viewed as ensuring that for every client in range of k or more towers, there always exists at least k different towers with which the client can communicate without interference.

As noted in Theorem 1, the number of edges in $(k-1)$-DG is at most ckn where $c = 3$ when the points are in general position and $c = 2$ when points are in convex position. This implies that the average degree of a vertex in $(k-1)$-DG is at most $2ck$ and, by using a standard argument which is omitted in this extended abstract, it can be seen that there are always *big* independent sets with bounded degree:

Lemma 4 *Every $(k-1)$-DG has an independent set of size at least $n/8ck$ where each vertex in the set has degree at most $4ck$.*

The coloring algorithm is simple and repeated applies the above lemma. Find a large independent set in the $(k-1)$-DG of the given point set P. Assign a unique color to the points in the independent set. Remove these points from P and repeat as long as $|P| > 0$. In the next lemma, we show that this algorithm provides a k-conflict free coloring and the total number of colors used is $\log n/\log(8ck/(8ck-1))$

Lemma 5 *With $\log n/\log(8ck/(8ck-1))$ colors, a set of n points can be colored so that every circle containing at least k points contains k points whose color is unique.*

Proof. First, at each iteration, we remove an independent set of size at least $n/8ck$. Let $C(n)$ represent the number of colors used for a $(k-1)$-DG graph with n vertices. We can bound $C(n)$ with the following recurrence: $C(n) \leq C((8ck-1)n/8ck) + 1$. This recurrence resolves to $C(n) \leq \log n/\log(8ck/(8ck-1))$ as required.

Next, we show that the coloring is k-conflict free. Let C be any circle containing a set P of at least k points. Consider the k points in C whose colors have highest value (recall that the first independent set was given color 0 and an independent set removed at step i was given color i). If all these k points have unique colors, the lemma is proved. For sake of a contradiction, assume that at least 2 of these k points have the same color. Let i be the largest color whose value is not unique. Note that there are fewer than k points in P whose color value is strictly greater than i. Also note that at iteration i of the algorithm, all points with color less than i have been removed from P. Let P_i be the set of points in P receiving color i. Since C contains P_i, there is a circle C' contained in C that has two points x, y of P_i on its boundary and no points of P_i in its interior. However, since there are fewer than k points whose color is larger than i, this means that C' contains fewer than k points in its interior at iteration i of the algorithm. However, this contradicts the fact that x and y are in an independent set selected at iteration i. $\square$

Corollary 6 *A set of n points in general position can be colored with $\log n/\log(24k/(24k-1))$ colors so that every circle containing at least k points contains k points whose color is unique. If the set of n points is in convex position, then $\log n/(\log(16k/(16k-1))$ colors are sufficient*

Note that we only used the fact that there are large numbers of vertices of bounded degree in $(k-1)$-DG in order to show that there is a sufficiently large independent set. If one can find a larger independent set that is guaranteed to exist in all $(k-1)$-DG graphs, then the above bounds can be improved.

6 Conclusion

In this work we have investigated some properties of higher order Delaunay graphs. There are several questions that remain open, and we emphasize the following:

- give some lower bound on the size of k-DG and a tight upper bound,
- show that k-DG is Hamiltonian for small k.

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