Transforming Spanning Trees and Pseudo-Triangulations

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Abstract

Let T_S be the set of all crossing-free straight line spanning trees of a planar n-point set S. Consider the graph \mathcal{T}_S where two members T and T' of T_S are adjacent if T intersects T' only in points of S or in common edges. We prove that the diameter of \mathcal{T}_S is $O(\log k)$, where k denotes the number of convex layers of S. Based on this result, we show that the flip graph \mathcal{P}_S of pseudo-triangulations of S (where two pseudo-triangulations are adjacent if they differ in exactly one edge – either by replacement or by removal) has a diameter of $O(n \log k)$. This sharpens a known $O(n \log n)$ bound. Let $\widehat{\mathcal{P}}_S$ be the induced subgraph of pointed pseudo-triangulations of \mathcal{P}_S . We present an example showing that the distance between two nodes in \mathcal{P}_S is strictly larger than the distance between the corresponding nodes in \mathcal{P}_S .

1 Introduction

Let S be a set of n points in general position in the plane (no three points of S are on a common line). We denote by T_S the set of all crossing-free straight line spanning trees of S. Several authors investigated the question of whether, and how fast, two members of T_S can be transformed into each other by means of predefined rules. Avis and Fukuda [5] considered the graph with node set T_S where two spanning trees are adjacent if they have all but one edge in common (i.e., differ by a single edge move). They showed that the diameter of this graph is at most 2n - 4. The impact of several more involved transformations, including length-reducing edge moves and so-called edge slides, has been studied in Aichholzer, Aurenhammer, and Hurtado [1]. Recently, Aichholzer and Reinhardt [4] proved that the edge slide distance between two trees in T_S is $O(n^2)$.

Define the graph $\mathcal{T}_S = (T_S, A)$ whose set of arcs A

consists of pairs of trees in T_S that intersect each other only in points of S or in common edges. In other words, A contains all pairs (T, T') such that no edge of T crosses any edge of T'. The arcs of \mathcal{T}_S correspond to rather powerful transformations, namely, the replacement of a tree by some 'compatible' tree. Not surprisingly, a bound of $O(\log n)$ on the diameter of \mathcal{T}_S is easily obtained. A core result in [1] states that \mathcal{T}_S contains a path of size $O(\log n)$ from every member of T_S to the minimum spanning tree of Ssuch that tree lengths decrease along this path.

In this note we prove that the diameter of \mathcal{T}_S is $O(\log k)$, where k is the number of convex layers L_1, \ldots, L_k of S. That is, L_1 is the boundary of the convex hull of S, and L_i is defined recursively as the boundary of the convex hull of $S \setminus \bigcup_{j < i} L_j$, for $2 \leq i \leq k$ and $k = \min\{i \mid L_{i+1} = \emptyset\}$. We do not know whether this bound is asymptotically tight. In particular, we do not have any example where the diameter of \mathcal{T}_S is not a constant.

Interestingly, the diameter of \mathcal{T}_S is related to flip distances in pseudo-triangulations. A pseudo-triangle is a planar polygon with exactly three interior angles less than π . A pseudo-triangulation of S is a partition of the convex hull of S into pseudotriangles whose vertex set is S. The flip graph of pseudo-triangulations, \mathcal{P}_S , has as its set of nodes all possible pseudo-triangulations of S. Two pseudotriangulations are connected in \mathcal{P}_S by an arc if they differ in exactly in one edge, either by replacement or removal. In other words, each arc of \mathcal{P}_S corresponds to an exchanging or a removing edge flip. Aichholzer, Aurenhammer, and Krasser [3] (see also [2]) proved that the diameter of \mathcal{P}_S is $O(n \log n)$. Let \mathcal{P}_S be the induced subgraph of elements of \mathcal{P}_S having exactly 2n-3 edges (the minimum number of edges a pseudotriangulation of S can have). The elements of $\widehat{\mathcal{P}}_S$ are called minimum, or pointed, pseudo-triangulations; each vertex of such a pseudo-triangulation is *pointed*, that is, all its incident edges lie in an angle less than π . Bereg [6] showed that the diameter of \mathcal{P}_S is still bounded by $O(n \log n)$, and in [2] it is proved that if the diameter of $\widehat{\mathcal{P}}_S$ is O(n) then the same is true for \mathcal{P}_S . On the other hand, the flip distance for triangulations (i.e. pseudo-triangulations having the maximum number of edges) is known to be $\Theta(n^2)$ in the worst case. In [8], Hurtado, Noy, and Urrutia refined the bound for triangulations to O(nk).

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For both graphs \mathcal{P}_S and $\widehat{\mathcal{P}}_S$ it is an open problem to determine tight asymptotic bounds on the diameter. We prove that the diameter of \mathcal{P}_S is $O(n \log k)$, using our result for the graph \mathcal{T}_S . In particular, if the diameter of \mathcal{T}_S is constant then the diameter of \mathcal{P}_S is $\Theta(n)$. We also demonstrate that the distance between certain nodes in $\widehat{\mathcal{P}}_S$ is strictly larger than the distance between the corresponding nodes in \mathcal{P}_S . A more comprehensive study of the diameters of \mathcal{T}_S and \mathcal{P}_S can be found in Huemer [7].

2 An upper bound on the diameter of T_S

We show that the diameter of \mathcal{T}_S can be related to the number k of convex layers of the point set S.

Observation 1 Let Δ be a triangulation of S. Let x be a point of S which lies on some layer L_i of S, for $i \geq 2$. Then Δ contains an edge xy that does not cross L_i and such that y lies on a layer L_j with j < i.

Proof. If such an edge does not exist then x must be a pointed vertex of Δ . But a triangulation does not contain pointed vertices, except on layer L_1 .

Theorem 1 Let a point set S be given whose number of convex layers is k. The diameter of \mathcal{T}_S is $O(\log k)$.

Proof. Define a *layer tree* to be a non-crossing spanning tree of S which contains all but one edge of each layer L_i of S and which connects consecutive layers (by single edges); Figure 1 gives an example. We show below that any given spanning tree $T \in T_S$ can be transformed into some layer tree using $O(\log k)$ transformations, as defined by the arcs of \mathcal{T}_S . This implies the theorem, because two layer trees clearly can be made to coincide by applying at most two transformations.

Consider some triangulation Δ that contains T. Due to Observation 1, for every point $x \in S \setminus L_1$ there is an edge in Δ to some layer with lower index. We select one such edge per point in $S \setminus L_1$, and in addition, all edges but one of L_1 . The selected edges constitute a new spanning tree, T', of S. As T' and Tlive in the same triangulation, a single transformation is capable of replacing T by T'.

For a point $x \in S$, let $g_k(x)$ be the shortest path from x to a point on L_k such that $g_k(x)$ does not cross T'. As no edge of T' crosses any layer twice, $g_k(x)$ visits points on layers with increasing index. On the other hand, by Observation 1, there is a path $g_1(x)$ from x to L_1 that does not cross T' and that visits points on layers with decreasing index. Now, for all points x on layers $L_1, \ldots, L_{k/2}$ take the path $g_1(x)$, and for all points x on layers $L_{k/2}, \ldots, L_k$ take the path $g_k(x)$. The union of all these paths with L_1 is a connected graph, G. By construction, G neither crosses the tree T' nor the layer $L_{k/2}$. We select



Figure 1: A layer tree for k = 3

from G a spanning tree, T'', which contains L_1 minus one edge. T' can be transformed into T'' in one step, and T'' can be transformed into a spanning tree T_1 that contains, in addition, $L_{k/2}$ minus one edge, in one more step.

In summary, after a constant number of transformations we arrive at two independent subproblems of size k/2. Therefore, with the same effort, we can transform the tree T_1 into a tree that contains, in addition to L_1 and $L_{k/2}$, from both layers $L_{k/4}$ and $L_{3k/4}$ all edges but one. We conclude that $O(\log k)$ transformations suffice to generate a layer tree for S.

3 Bounding the diameter of \mathcal{P}_S

Next we show that an upper bound on the diameter of \mathcal{T}_S also gives an upper bound on the diameter of the flip graph \mathcal{P}_S of pseudo-triangulations. We make use of a lemma from [3] on flip distances in simple polygons.

Lemma 2 Let Q be a simple polygon with m edges. The flip distance between any two triangulations of Q is O(m), if exchanging, removing, and inserting edge flips are allowed.

Theorem 3 Let a set S of n points be given. If the diameter of \mathcal{T}_S is d then the diameter of \mathcal{P}_S is O(nd).

Proof. Every pseudo-triangulation of S can be completed to a triangulation by applying O(n) inserting edge flips. It thus suffices to show that any two triangulations Δ_1 and Δ_2 are connected in \mathcal{P}_S by a sequence of O(nd) flips. Let Δ_1 and Δ_2 contain spanning trees T_1 and T_2 of S, respectively. There is a path of length d in \mathcal{T}_S which connects T_1 and T_2 . We show that for consecutive trees T and T' on this path, the distance in \mathcal{P}_S between any triangulation Δ containing T and any triangulation Δ' containing T' is O(n). Let us 'cut' the triangulation Δ along the edges of T. This gives several triangulated polygons. Note that no edge of T' crosses any edge of such a polygon, because T' and T are adjacent in \mathcal{T}_S . By Lemma 2, we can modify the triangulation within each such polygon Q_i so as to contain all the edges of T' within Q_i in $O(m_i)$ flips, if Q_i has m_i edges. Thus Δ can be transformed into a triangulation Δ'' that contains T'in $O(\sum m_i)$ flips. This sum is bounded by n+2(n-1), counting the edges of L_1 plus two times the edges of T. Similarly, in a second step, we cut Δ'' along T' and transform Δ'' into the desired triangulation Δ' using another O(n) flips. \Box

Corollary 4 The diameter of the flip graph \mathcal{P}_S is bounded by $O(n \log k)$, where k is the number of convex layers of S.

Lemma 2 also holds for pointed pseudo-triangulations [3]. Thus, we are also interested in the graph $\widehat{\mathcal{T}}_{S}$ of pointed spanning trees of S, where two trees are adjacent if there exists a pointed pseudotriangulation which contains them both. If we can bound the diameter of $\widehat{\mathcal{T}}_{S}$ by d then the diameter of $\widehat{\mathcal{P}}_{S}$ is O(nd), applying the argumentation of Theorem 3. Moreover, the bound O(nd) carries over to the flip graph \mathcal{P}_{S} by a result in [2].

4 Comparing distances in \mathcal{P}_S and $\widehat{\mathcal{P}}_S$

In a pointed pseudo-triangulation the number of edges is minimum. This suggests that this type of pseudotriangulation is most flexible as far as adaption by flips is concerned. In other words, one might conjecture that the distance between two nodes of \mathcal{P}_S does not increase when we are required to stay within the subgraph $\widehat{\mathcal{P}}_S$ of \mathcal{P}_S . In the following we present an example that refutes this conjecture. Assume n > 9in the following.

The underlying set S of n points is shown in Figures 2 and 3. It consists of three subsets $P = \{p_1, ..., p_{n/3}\}, Q = \{q_1, ..., q_{n/3}\}, \text{ and } R = \{r_1, ..., r_{n/3-1}\}$ in convex position, and one interior point m. The last point is chosen to lie to the left of $p_1q_{n/3}$, of $q_1r_{n/3-1}$, and of $r_1p_{n/3}$. Two pointed pseudo-triangulations PT_1 (Figure 2) and PT_2 (Figure 3) are drawn on S.

Lemma 5 The distance between PT_1 and PT_2 in $\widehat{\mathcal{P}}_S$ is at least n-3.

Proof. Every pointed pseudo-triangulation of S has exactly 2n - 3 edges. PT_1 and PT_2 have the n/3 - 1edges r_im in common, plus the n - 1 edges of the convex hull of S. Thus 2n/3 - 1 edges are different. We will show that to obtain the very first edge of PT_2 that is not in PT_1 , at least n/3 - 1 (exchanging)



Figure 2: The pseudo-triangulation PT_1

edge flips are required. Then, when the first edge is present, there still remain at least 2n/3 - 2 different edges. For each such edge, at least one flip is needed in addition. This gives a lower bound of n - 3 flips.

 PT_1 contains edges mq_i and q_1p_i which must be transformed into edges $p_{n/3}q_i$ and mp_i of PT_2 . If the first edge of PT_2 that is created is of type $p_{n/3}q_i$ then this edge crosses n/3 - 1 edges q_1p_i . All these edges must be replaced beforehand. If, otherwise, the first edge of PT_2 that is created is of type mp_i then either all edges mq_i or all edges mr_i must be replaced first, because the point m has to stay pointed. So, in any case, at least n/3 - 1 flips are necessary to obtain the first edge of PT_2 .

Lemma 6 The distance between PT_1 and PT_2 in \mathcal{P}_S is at most 2n/3.

Proof. We construct a sequence of 2n/3 flips that transforms PT_1 into PT_2 . The edge flip that inserts the edge mp_1 into PT_1 is applied first. Now edge q_1p_i is flipped into edge mp_{i+1} by an exchanging flip, for $i = 1, \ldots, n/3 - 1$. Next, edge mq_i is exchanged by edge $p_{n/3}q_{i+1}$, for $i = 1, \ldots, n/3 - 1$. Finally, we apply the edge flip that removes $mq_{n/3}$ and gives PT_2 . \Box



Figure 3: The pseudo-triangulation PT_2

Corollary 7 There exist pointed pseudo-triangulations PT_1 and PT_2 whose distance in the graph $\widehat{\mathcal{P}}_S$ can only be realized by a flip sequence that affects edges common to PT_1 and PT_2 .

Proof. To see this, let the subset R in Figures 2 and 3 consist of a single point r_1 . Then the edge mr_1 is common to PT_1 and PT_2 . If flipping mr_1 is not allowed then at least n-3 flips are needed to transform PT_1 into PT_2 , by the same arguments as in the proof of Lemma 5. Otherwise, we change mr_1 (in PT_1) to the edge mp_1 first. Then we apply the same sequence of 2n/3 - 2 exchanging flips as in the proof of Lemma 6. Finally, $mq_{n/3}$ is changed to mr_1 which gives PT_2 . This sequence consists of only 2n/3 flips.

Note that the reduction in flip distance in Lemma 6 and Corollary 7, respectively, stems from creating the edge mp_1 . This edge is outruled in Lemma 5 by the required pointedness of m, and in Corollary 7 by being the result of flipping the common edge mr_1 .

5 Conclusion and open problems

We gave a bound on the diameter of the graph \mathcal{T}_S of non-crossing spanning trees and related this result to transforming pseudo-triangulations. The problem of bounding the diameter of \mathcal{T}_S is also of interest on its own. So far we were not able to find two non-crossing spanning trees on the same point set S whose distance in \mathcal{T}_S is more than constant.

Conjecture 1 The diameter of \mathcal{T}_S is sublogarithmic.

We also restate the well known problem of determining the diameter of the flip graph \mathcal{P}_S of pseudotriangulations.

Conjecture 2 The diameter of \mathcal{P}_S is $\mathbf{o}(n \log n)$.

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