Discrete Curvatures and Gauss Maps for Polyhedral Surfaces

Lyuba Alboul* Gilberto Echeverria† Marcos Rodrigues‡

Abstract

The paper concerns the problem of correct curvatures estimates directly from polygonal meshes. We present a new algorithm that allows the construction of unambiguous Gauss maps for a large class of polyhedral surfaces, including surfaces of non-convex objects and even non-manifold surfaces. The resulting Gauss map provides shape recognition and curvature characterisation of the polyhedral surface (polygonal mesh) and can be used further for optimising the mesh or for developing subdivision schemes.

1 Introduction

In many applications a physical object is represented by discrete data, obtained by some measurement system. A polyhedral model (a triangular mesh, piecewise linear surface) is the easiest way to obtain a preliminary sketch of the given object. A solid object is represented by its boundary, i.e. by the surface that bounds the object. Triangular or polygonal meshes are commonly used in modern computer-related applications to represent surfaces in three-dimensional space. Therefore, there is a substantial need for accurate estimates of geometric attributes such as surface area, normal vectors, and curvatures directly from a mesh. A smooth surface S is uniquely characterised and quantified by the *metric tensor* and by the *Weingarten map* or the *shape operator* [Kuhn02]. The shape operator is the second-order invariant (in other words, curvature) that completely determines the shape of the surface S. In recent years significant efforts have been made to define the analogues of differential geometry concepts on meshes that imitate those of a smooth surface ([Alb96, Bor03, Dyn01, Malt02, Mey03]). Among those concepts surface curvatures are most important. Surface curvatures are basic measures to describe the local shape of a smooth surface. However, a mesh (a polyhedral surface) is not smooth, and there is still no consensus about the most appropriate way of estimating such geometric quantities as curvatures. On the other hand, methods are being developed to capture curvature information without referring to higher-order formulas of differential geometry. These methods are based on the discrete curvature concepts and are of growing interest for geometric modelling [Malt02, Alb03, Dyn01, Mey03]. Discrete curvatures can be computed directly from the polygonal mesh. The principal difference to smooth surfaces is that the curvatures in polyhedral surfaces are concentrated around the vertices and along the edges.

If we think of a polyhedral surface as an approximation of a smooth surface, then, informally speaking, curvatures of a domain of the underlying smooth surface are 'glued' together in the corresponding domain of a polyhedral surface.

Therefore, analogue measures of curvature in a piecewise linear setting should be analogues of *integral formulae* for curvature in a 'smooth' setting and should preserve integral relations for curvature, such as the Gauss-Bonnet theorem ([Br79, Banch70, Alb96]. Such analogues exist and were introduced long ago in the frames of the theory of non-regular surfaces (see an overview in [Alb96]). These analogues were discussed in detail in [Br79], where the authors also compare discrete curvatures with their smooth counterparts.

In the last five-six years the amount of papers that explore discrete curvatures in one or another context has increased significantly. Much attention is paid to the discrete Gaussian curvature, known also as the *angle deficit*. The angle deficit is used to estimate the Gauss curvature of smooth surfaces. In [Bor03] the problem of the correct estimation of the Gauss curvature is investigated in detail, and they show that approaches based on the use of normalized angular deficits are often erroneous, and can be applied correctly only if the geometry of meshes is precisely controlled. We agree with them, and in this paper we highlight why the angular deficit is neither sufficient to estimate the Gaussian curvature of the underlying smooth surface nor to capture the curvature information of a polyhedral surface. Loosely speaking, the reason is that there are more curvatures for polyhedral surfaces than for smooth ones [Br79, Alb96, Alb03]. This fact is still not fully acknowledged in geometric applications, but without addressing it, it is impossible to develop correct curvature estimates.

Besides the need to derive correct curvature estimates directly from polygonal meshes, there is also a need for visualisation of an object in order to explore complex shapes and emphasize hidden details. We pro-

*Materials and Engineering Research Institute, Sheffield Hallam University, UK, L.Alboul@shu.ac.uk

†idem, G.Echeverria@shu.ac.uk

‡idem, M.Rodrigues@shu.ac.uk

pose an approach that addresses both needs, and that empowers us to correctly and consistently describe and visualise complex 3D shapes based on curvature properties. Our method to characterise surface shape is based on constructing the Gauss map directly from the polygonal mesh, an area of research with still scarce and ambiguous results for non-convex objects [Low02]. The resulting Gauss map provides a description of the surface by determining its domains with respect to incorporated curvatures. Each domain can be split up into uniquely determined sub-domains; therefore each surface can be associated with the introduced *Gauss map signature*, abbreviated as GMS. The GMS extracts convex, concave and saddle regions in the underlying surface. These regions are often only implicitly present in a polyhedral surface, and cannot be determined by the sign of the angle deficit only. The GMS method besides shape recognition and description can be used for optimisation of the underlying model or for developing subdivision schemes. The method provides also a better insight into the geometric structure of complex triangle meshes, by describing various vertex types, some of them with a very complex GMS. A good understanding of the geometry of meshes is a step towards more robust mesh manipulation algorithms. Finally, the proposed GMS method is simple to compute, easy to view dynamically and effective in visualising complex polyhedral surfaces.

2 Polyhedral Surfaces: Discrete curvatures and Gauss map

By a polyhedral surface we understand a triangulated polyhedral surface. Designating $\mathbf{V}$ as a finite point set in three-dimensional space, $\mathbf{V} = \{V_i, i = 1, \ldots, n\}$, we denote by $\mathbf{P}(\mathbf{V})$ a polyhedral surface with the vertex set $\mathbf{V}$. The term *polyhedron* refers to a closed polyhedral surface. In such a setting a polyhedron is bounded, but might be non-simple, i.e. non-homeomorphic to a sphere, as well as being multi-connected and self-intersecting, and its interior volume is not necessarily part of the polyhedron. Therefore, a polyhedron is not necessarily a solid body. Given a polyhedron $\mathbf{P}(\mathbf{V})$, the set of its vertices is denoted by V, the edges by E, and the faces by F.

Definition 1 *The star* $\mathbf{Str}(\nu)$ *of a vertex* ν *is the union of all the faces and edges that contain the vertex, and the link* $\mathbf{Lnk}(\nu)$ *of the star (the boundary of the star) is the union of all those edges of the faces of the star* $\mathbf{Str}(\nu)$ *that are not incident to* ν.

2.1 Discrete Curvatures

In this paper we are interested only in discrete curvatures related to the integral Gaussian curvature, i.e. those that are supported on the vertices. The common expression for the integral Gaussian curvature of a domain U of a smooth surface S is $\int_U K dA$ [Kuhn02]. **Curvature ω around vertex ν** is defined as:

$$\omega = 2\pi - \theta \tag{1}$$

where $\theta = \sum \alpha_i$ is the total angle around vertex ν, and α_i are those angles of the faces in the $Str(\nu)$ that are incident to ν. Sometimes one refers to ω simply as the Gaussian curvature around the vertex, or *discrete Gaussian curvature*. Obviously, expression 1 is valid for any point $x \in P$. For a domain $U \subseteq P$ the total curvature Ω_U is determined as $\Omega_U = \sum_{\nu \in U} \omega_\nu$. For an oriented closed polyhedral surface P of genus g Ω_U is equal to $(1-g)4\pi$, so the discrete analogue of the Gauss-Bonnet theorem is preserved.

Positive (extrinsic) curvature ω^+: The following measure which we determine is an analogue of the total absolute curvature of a polyhedral domain. However, in Figure 1 we can see that in both polyhedra all curvatures ω_ν are positive and actually are equal for every corresponding vertex.

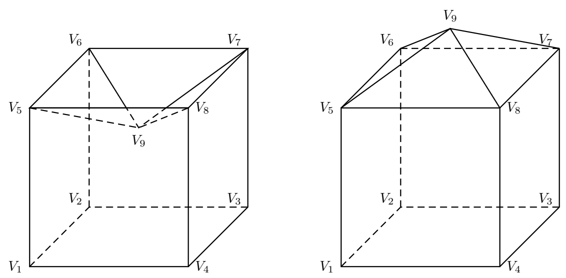


Figure 1: Two polyhedra

Therefore, we have:

$$\Omega(P_1) = \Omega(P_2) = \sum_{\nu \in P_1} |\omega_\nu| = \sum_{\nu \in P_2} |\omega_\nu| = 4\pi. \tag{2}$$

The left polyhedron is non-convex, but the above equation does not reflect this fact. For a closed non-convex smooth surface S the total absolute curvature $K_{abs} = \int_S |K| dA$ is greater than 4π; therefore, $\sum_{\nu \in P} |\omega_\nu|$ is not an appropriate analogue of K_{abs}. The problem is that the curvature ω around a vertex may consist of positive and negative 'parts' that are 'glued' together; and the task is to separate them. If vertex ν belongs to the boundary of the convex hull of its star (i.e. the convex hull of ν and all vertices in its star), then we can single out another star $\mathbf{Str}^+(\nu)$ with ν as the vertex and those edges of $\mathbf{Str}(\nu)$ that belong to the boundary of the convex hull. The edges of $\mathbf{Str}^+(\nu)$ will determine the faces of $\mathbf{Str}^+(\nu)$. We refer to $\mathbf{Str}^+(\nu)$ as the convex cone of vertex ν. Then

$$\omega^+ = 2\pi - \theta^+ \tag{3}$$

where θ^+ is the total angle around ν in $\mathbf{Str}^+(\nu)$. ω^+ is equal to zero, if the vertex and all the vertices in its star lie in the same plane. If the convex cone around ν doesn't exist, i.e. ν lies inside the convex hull of $\mathbf{Str}(\nu)$, then ω^+ is, by definition, equal to zero.

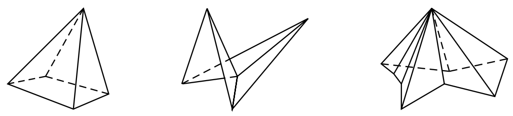
Negative (extrinsic) curvature ω^-: We can now 'extract' the negative part of ω as follows

$$\omega^- = \omega^+ - \omega \tag{4}$$

Absolute (extrinsic) curvature ω_{abs}:

$$\omega_{abs} = \omega^+ + \omega^- \tag{5}$$

On the basis of the types of curvatures around a vertex one distinguishes three basic types of vertices for an embedded polyhedral surface ([Br79, Alb96]): convex vertices ($\omega^+ = \omega$), saddle vertices ($\omega^- = -\omega$) and mixed vertices ($\omega^+ > 0, \omega^+ \neq \omega$) (see Figure 2).



(i) (ii) (iii)

Figure 2: Examples of convex (i), saddle (ii), and mixed (iii) vertices

A mixed vertex, however, possesses always the convex cone around its star. A mixed vertex and its correspondent convex cone are shown in Figure 3.

(i) (ii)

Figure 3: Mixed vertex (i) and its convex cone (ii)

Total absolute extrinsic curvature Ω_{abs}: is defined as the sum of absolute extrinsic curvatures of all the vertices of a polyhedral surface P:

$$\Omega_{abs} = \sum_i \omega_{abs}(\nu_i) = \sum \left[\omega^+(\nu_i) + \omega^-(\nu_i)\right] \tag{6}$$

Ω_{abs} takes different values on the polyhedra that are depicted in Figure 1. It is equal to 4π on the right polyhedron, as it represents a convex body, and is greater than 4π on the left polyhedron.

2.2 Gauss map

Separation of the positive and negative parts of the curvature for a mixed vertex can also be carried out using the Gauss map. For a domain U of smooth surface S the Gauss map $N(U)$ may be thought of as the map assigning to each point $p \in U$ the point on the unit 2-sphere $S^2 \in \mathbf{R}^3$, by 'translating' the unit normal vector $N(p)$ to the origin [Kuhn02]. The endpoints of normals, therefore, will cover a certain region on S^2. If a neigbourhood $U(p)$ is small such that the map $N(U(p))$ is one-to-one and orientation-preserving (outward normals at corresponding points on S and S^2 correspond), then the area $N(U(p))$ is considered positive, and the corresponding region $U(p)$ is said to be strictly *convex* and the Gaussian curvature at p defined as $|K(p)| = \lim_{U(p)\downarrow p} \frac{AreaN(U(p))}{AreaU(p)}$, is positive, i.e. $K(p) > 0$. If the map $N(U(p))$ is one-to-one but orientation reversing, then the area $N(U(p))$ is considered to be negative, p is a saddle point and $K(p) < 0$. Of course, different regions of S can be mapped to the same region on the unit sphere, which results in multiplicities of the Gauss map.

To compute directly the image of the Gauss map of a given vertex, we need to construct the outward vector normal for each of the facets around a vertex and then draw geodesics arcs between the images of neighbouring faces to obtain a graphic image. The union of the Gauss maps for all vertices is the Gauss map of a polyhedral surface.

An orientation of the contour around the vertex on a polyhedral surface induces the orientation on the boundary of the spherical polygon. Thus we can evaluate the curvature around the vertex by computing the area of the spherical image with the sign + (plus) in the case that the orientation is preserved, and with the sign − (minus) otherwise.

3 Results

To characterise a polyhedral model we have developed algorithms that have the following functionalities:

1. *Determination of the Gauss Map* for each of the vertices in V; and
2. *Curvature Visualisation*, which displays a graphical representation of the Gauss map.

We are able to divide the Gauss Map for a vertex into different spherical polygons, determine the orientation of each polygon and thus its sign. Therefore, we are able not only to separate $\omega(v)$ for a vertex v into positive and negative parts $\omega^+(v)$ and $\omega^-(v)$, but to separate into subparts of the same sign. The number of subparts together with their signs represents the *Gauss map signature* of a vertex. Each subpart of the negative sign represents a potential (hidden) saddle region.

The main advantage of our method is that it allows the determination of incorporated curvatures of various types of vertices, including all the abovementioned ones and much more complex such as the *monkey saddle*, or vertices with *self-intersections*,

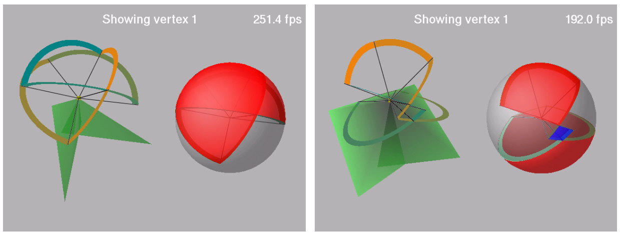


Figure 4: Pinch vertex and a reverse pinch vertex with corresponding Gauss maps

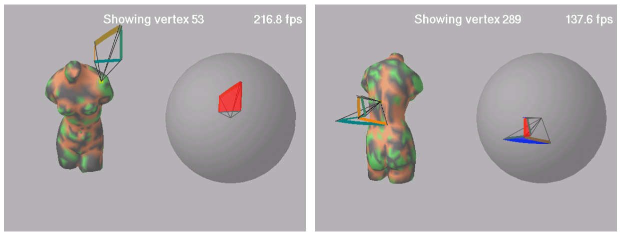


Figure 5: Torso and its Gauss map visualisations

which don't fit exactly in the category 'mixed', described in the previous section. Eventually, we can determine the curvatures of a vertex of any type (of an oriented polyhedral surface P). It is also possible to display the Gauss Map for all the vertices of the object simultaneously, or select only one of the vertices for its Gauss Map to be shown exclusively (or, correspondingly, to visualise the Gauss map of a region on the surface). The method is interactive, and we can visualise the regions of positive curvature separately from the regions of negative curvature.

Examples of Gauss map visualisations are given below. The display of the Gauss Map is done in two different views, or scenes, and is implemented using OpenGl. The left scene shows the model of the original object and, in the right scene, the areas for the Gauss Map are drawn on top of a sphere. Positive areas are shown in red, while negative areas are displayed in blue (lighter and darker grey in the black-white print). The corresponding areas on the object are coloured in green and red respectively (dark and light grey in the black-white print).

Figure 4 shows the Gauss map visualisation of two complex vertices with self-intersections, which we call a *pinch vertex* and a *reverse pinch vertex*. In order to understand the difference between a pinch vertex and a reverse pinch vertex, imagine a walk along the link of the star of a vertex ν. In the case of the pinch vertex the walk makes two full turns around the vertex, both turns have the same orientation (for example, counter clockwise). In the case of the reverse pinch point, the walk makes also two full turns, one is, for example, in the counter clockwise direction, and the second one - in the 'reverse' direction (i.e. clockwise). The Gauss map of the pinch vertex has two overlapping areas, each of positive sign. One area is equal to the curvature of the convex star of the pinch vertex. The Gauss map of the reverse pinch vertex has also two areas of positive curvature, separated by the area of negative curvature.

A more complex object and its Gauss map visualisation are presented in Figure 5.

4 Future work

Current on-going research includes the visualisation of the processes of mesh simplification and optimisation by using the GMS method, as well as to use it for developing subdivisions schemes based on curvature estimations.

References

[Alb96] Alboul, L. and van Damme, R. Polyhedral metrics in surface reconstruction. In: Mullineux, G. (ed.), *The Mathematics of Surfaces VI*, pp. 171-200, Oxford, 1996.

[Alb03] Alboul, L., Optimising triangulated polyhedral surfaces with self-intersections. In: Wilson, M. J., Martin, R. R., *Mathematics of Surfaces*, LNCS 2768, pp. 48-72, 2003.

[Banch70] Banchoff, T.F., Critical points and curvature for embedded polyhedral surfaces. *Amer. Math. Monthly* 77, pp. 475-485, 1970.

[Bor03] Borelli, V., Cazals, F, and Morvan J.-M., On the angular defect of triangulations and the pointwise approximation of curvature, *Comp. Aided Geom. Design* 20, pp. 319-341, 2003.

[Br79] Brehm, U., and Kühnel, W., Smooth approximation of polyhedral surfaces with respect to curvature measures. In: *Global differential geometry*, pp. 64-68, 1979.

[Dyn01] Dyn N. et al., Optimising 3D triangulations using discrete curvature analysis. In: Lyche, T., and Schumaker, L. L. (eds.), *Mathematical Methods in CAGD*, pp. 1-12, Vanderbilt University Press, 2001.

[Kuhn02] Kühnel, W., Differential geometry. Curves-Surfaces-Manifolds, *Amer Math Society*, 2002.

[Low02] Lowekamp, B., Rheingans, P., and Yoo, T.S., Exploring surface characteristics with interactive Gaussian images: a case study. *Proc. of the conference on Visualization '02*, pp. 553-556, 2002.

[Malt02] Maltret, J.-L., and Daniel, M., Discrete curvatures and applications: a survey. *Research report LSIS-02-004*, March 2002.

[Mey03] Meyer, M., Desbrun, M., Schröder, P., and Barr, A. H., Discrete differential-geometry operators for triangulated 2-manifolds. In: Hege, H.-Ch., and Polthier, K. (eds.), *Visualization and Mathematics III*, pp. 35-59, Springer-Verlag, 2003.