Improved Lower Bound on the Geometric Dilation of Point Sets[*]

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Abstract

Let G be an embedded planar graph whose edges are curves. The *detour* between two points p and q (on edges or vertices) of G is the length of a shortest path connecting p and q in G divided by their Euclidean distance $|pq|$. The maximum detour over all pairs of points is called the *geometric dilation* $\delta(G)$. Ebbers-Baumann, Grüne and Klein have shown that every finite point set is contained in a planar graph whose geometric dilation is at most 1.678, and some point sets require graphs with dilation $\delta \geq \pi/2 \approx 1.57$. They conjectured that the lower bound is not tight. We use new ideas, a disk packing result and arguments from convex geometry, to prove this conjecture. The lower bound is improved to $(1 + 10^{-11})\pi/2$.

1 Introduction

Consider a planar graph G embedded in $\mathbb{R}^2$, whose edges are curves[1] that do not intersect. Such graphs arise naturally in the study of transportation networks, like waterways, railroads or streets. For two points, p and q (on edges or vertices) of G, the *detour* between p and q in G is defined as

$$\delta_G(p, q) = \frac{d_G(p, q)}{|pq|}$$

where $d_G(p, q)$ is the shortest path length in G between p and q and $|pq|$ denotes the Euclidean distance, see Figure 1 for an example.

Good transportation networks should have small detour values. In a railroad system, access is only possible at stations, the vertices of the graph. Hence, to measure its quality we can take the maximum detour over all pairs of vertices. This results in the well-known concept of *graph-theoretic dilation* studied extensively in the literature on spanners, see [7] for a survey.

[*]There is a full version [3] of this paper available.

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[1]For simplicity we assume here that the curves are piecewise continuously differentiable, but think that the proofs can be extended to a broader class of curves.

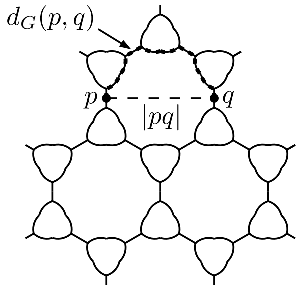

Figure 1: A grid G of small dilation $\delta(G) = \delta_G(p, q) = d_G(p, q)/|pq| < 1.678$ introduced in [5]

However, if we consider a system of urban streets, houses are usually spread everywhere along the streets. Hence, we have to take into account not only the vertices of the graph but all the points on its edges. The resulting supremum value is the *geometric dilation*

$$\delta(G) := \sup_{p,q \in G} \delta_G(p, q) = \sup_{p,q \in G} \frac{d_G(p, q)}{|pq|}$$

on which we concentrate in this article. Several papers [6, 13, 2] have shown how to efficiently compute the geometric dilation of polygonal curves. Besides this the geometric dilation was studied in differential geometry and knot theory under the notion of distortion, see e.g. [9, 12].

Ebbers-Baumann et al. [5] recently considered the problem of constructing a graph of lowest possible geometric dilation containing a given finite point set on its edges. Even for three given points this is a difficult task. Therefore they started by providing an upper and a lower bound on the dilation necessary to embed any finite point set, i.e. on the value

$$\Delta := \sup_{P \subset \mathbb{R}^2,\ P \text{ finite}} \ \inf_{G \supset P,\ G \text{ finite}} \delta(G) \ .$$

They showed that a slightly perturbed version of the grid in Figure 1 can be used to embed any finite point set. Thereby they proved $\Delta < 1.678$.

They also derived that $\Delta \geq \pi/2$, by showing that a graph G has to contain a cycle to embed a certain point set P_5 with low dilation, and by using that the dilation of every closed curve[2] C is bounded by $\delta(C) \geq \pi/2$.

[2]In this paper we use the notions "cycle" and "closed curve" synonymously.

They conjectured that this lower bound is not tight. It is known that circles are the only cycles of dilation $\pi/2$, see [4, Corollary 23], [1, Corollary 3.3], [12], [9]. And intuition suggests that one cannot embed complicated point sets with small dilation if every face of the graph has to be a circular disk. This idea would have to be formalized and still does not rule out that every point set could be embedded with dilation arbitrarily close to $\pi/2$. New ideas are needed to prove $\Delta > \pi/2$.

In Section 2 we show that cycles with dilation *close* to $\pi/2$ are *close* to circles, in some well-defined sense (Lemma 4). The lemma can be seen as an instance of a *stability result* for the geometric inequality $\delta(C) \geq \pi/2$, see [8] for a survey. Such results complement geometric inequalities (like the isoperimetric inequality between the area and the perimeter of a planar region) with statements of the following kind: When the inequality is fulfilled "almost" as an equation, the object under investigation is "close" to the object or class of objects for which the inequality is tight. An important idea in the proof of this stability result is a decomposition of any closed curve C into the two cycles C^* and M.

In Section 3 we use Lemma 4 to relate the dilation problem to a certain problem of packing and covering the plane by disks. By this we prove our main result $\Delta \geq (1 + 10^{-11})\pi/2$.

2 Result for Closed Curves

We want to prove that a simple closed curve C of low dilation is close to being a circle. We assume that C is given by an arc-length parameterization $c(t)$, $0 \leq t \leq |C|$, where $|C|$ denotes the length of C. Two points $p = c(t)$ and $\hat{p} = c(t \pm \frac{|C|}{2})$ on C that divide the length of C in two equal parts form a *halving pair* of C. The segment which connects them is a *halving chord*, and its length is the *halving distance*. We write $h = h(C)$ and $H = H(C)$ for the *minimum* and *maximum halving distance* of C.

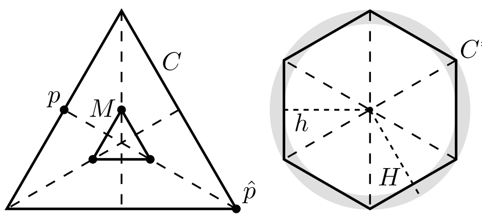

Figure 2: An equilateral triangle C, a halving pair $(p, \hat{p})$ and the derived curves C^* and M

To show that C is close to a circle, we consider a decomposition into two curves, see Figure 2 for an illustration. The *midpoint cycle* M is the cycle formed by the midpoints of the halving chords of C, and is given by the parameterization

$$m(t) := \frac{1}{2}\left(c(t) + c(t + \tfrac{|C|}{2})\right).$$

The curve C^* defined by

$$c^*(t) := \frac{1}{2}\left(c(t) - c(t + \tfrac{|C|}{2})\right)$$

is the result of the *halving pair transformation* defined in [4]. We get it by moving the midpoint of every halving chord to the origin. By definition, $c^*(t) = -c^*(t + \frac{|C|}{2})$, hence C^* is centrally symmetric. On the other hand, we have $m(t) = m(t + \frac{|C|}{2})$, and thus, M is traversed twice when C and C^* are traversed once. We define $|M|$ as the length of the curve M corresponding to one traversal.

The curve C^* has the same set of halving distances as C; thus, $h(C^*) = h(C) = h$ and $H(C^*) = H(C) = H$.

We have decomposed C into two components, from which it can be reconstructed:

$$c(t) = m(t) + c^*(t), \quad c\left(t + \tfrac{|C|}{2}\right) = m(t) - c^*(t) \quad (1)$$

To show that C is close to a circle, we first show that H/h is close to 1, i.e. C^* is close to a circle. Then, we prove that the length of the midpoint cycle is small. Combining both statements will deliver the desired result.

We use the following lemma to find an upper bound on the ratio H/h. Ebbers-Baumann et al. [4] have proved it for convex cycles using arguments from convex geometry similar to the ones in [10] but it can easily be extended to the non-convex case.

Lemma 1 *The geometric dilation $\delta(C)$ of any closed curve C satisfies*

$$\delta(C) \geq \arcsin\left(\frac{h}{H}\right) + \sqrt{\left(\frac{H}{h}\right)^2 - 1}.$$

Note that the function $g(x) = \arcsin 1/x + \sqrt{x^2 - 1}$ appearing on the right-hand side starts from $g(1) = \pi/2$ and is increasing on $[1, \infty)$. Approximating it by its Taylor expansion, we can show that $H/h \leq 1 + O(\varepsilon^{\frac{2}{3}})$ if $\delta(C) \leq (1 + \varepsilon)\frac{\pi}{2}$.

We still need an upper bound on the length $|M|$ of the midpoint cycle. We use the following lemma, which we think is of independent interest.

Lemma 2

$$4|M|^2 + |C^*|^2 \leq |C|^2.$$

Proof. Using the linearity of the scalar product and $|\dot{c}(t)| = 1$, we obtain from (1)

$$\begin{aligned}\langle \dot{m}(t), \dot{c}^*(t)\rangle &= \frac{1}{4}\left\langle \dot{c}(t) + \dot{c}(t + \tfrac{|C|}{2}), \dot{c}(t) - \dot{c}(t + \tfrac{|C|}{2})\right\rangle \\ &= \frac{1}{4}\left(|\dot{c}(t)|^2 - |\dot{c}(t + \tfrac{|C|}{2})|^2\right) = \frac{1}{4}(1 - 1) = 0\,.\end{aligned}$$

This means that the derivative vectors $\dot{c}^*(t)$ and $\dot{m}(t)$ are always orthogonal, thus $|\dot{m}(t)|^2 + |\dot{c}^*(t)|^2 = |\dot{c}(t)|^2 = 1$. This implies

$$\begin{aligned} |C| &= \int_0^{|C|} \sqrt{|\dot{m}(t)|^2 + |\dot{c}^*(t)|^2}\,\mathrm{d}t \\ &\geq \sqrt{\left(\int_0^{|C|} |\dot{m}(t)|\,\mathrm{d}t\right)^2 + \left(\int_0^{|C|} |\dot{c}^*(t)|\,\mathrm{d}t\right)^2} \\ &= \sqrt{4|M|^2 + |C^*|^2} \end{aligned}$$

The above inequality can be seen by a geometric argument. The left integral is the length of the curve $\gamma(s) := \left(\int_0^s |\dot{m}(t)|\,\mathrm{d}t, \int_0^s |\dot{c}^*(t)|\,\mathrm{d}t\right)$, while the right expression equals the distance of its end-points $\gamma(0) = (0,0)$ and $\gamma(|C|)$. □

Lemma 3 *If* $\delta(C) \leq (1+\varepsilon)\frac{\pi}{2}$, *then* $|M| \leq \frac{\pi h}{2}\sqrt{2\varepsilon + \varepsilon^2}$.

Proof. Because the dilation of C is at least the detour of a halving pair attaining minimum distance h, we get $(1+\varepsilon)\pi/2 \geq \delta(C) \geq |C|/2h$, implying

$$|C| \leq (1+\varepsilon)\pi h. \qquad (2)$$

If $|c^*(t)| < h/2$ held for any t, then, due to the central symmetry of C^*, the points $c^*(t)$ and $-c^*(t)$ would form a halving pair of distance $< h$, a contradiction. Hence, C^* encircles but does not enter the open disk $B_{h/2}(0)$ of radius $h/2$ centered at the origin 0. It follows

$$|C^*| \geq \pi h. \qquad (3)$$

By plugging everything together, we get

$$\begin{aligned} |M| &\overset{\text{Lemma 2}}{\leq} \frac{1}{2}\sqrt{|C|^2 - |C^*|^2} \\ &\overset{(2),(3)}{\leq} \frac{1}{2}\pi h\sqrt{(1+\varepsilon)^2 - 1} = \frac{\pi h}{2}\sqrt{2\varepsilon + \varepsilon^2}\,, \end{aligned}$$

which concludes the proof of Lemma 3. □

It should be intuitively clear (remember Figure 2) that the upper bound on H/h from Lemma 1 and the upper bound on $|M|$ of Lemma 3 imply that the curve C is contained in a thin ring if its dilation is close to $\frac{\pi}{2}$. This is the idea behind the omitted proof of the following lemma. We say that a cycle C is *enclosed in an* $(1+\varepsilon)$*-ring* if there is a radius $r > 0$ and a center $c \in \mathbb{R}^2$ such that the open region R bounded by C satisfies $B_r(c) \subseteq R \subseteq B_{(1+\varepsilon)r}(c)$.

Lemma 4 *Let* $C \subset \mathbb{R}^2$ *be any simple closed curve with dilation* $\delta(C) \leq (1+\varepsilon)\pi/2$ *for* $\varepsilon \leq 0.0001$. *Then* C *can be enclosed in a* $(1 + 3\sqrt{\varepsilon})$*-ring.*

By a special cycle C we can also show that this result cannot be improved apart from the coefficient of $\sqrt{\varepsilon}$. The lemma can be extended to a larger, more practical range of ε, by increasing the coefficient of $\sqrt{\varepsilon}$.

3 New Lower Bound

We will combine Lemma 4 with a disk packing result. A (finite or infinite) set $\mathcal{C}$ of disks in the plane with disjoint interiors is called a *packing*.

Theorem 5 (Kuperberg, Kuperberg, Matoušek and Valtr [11]) *Let* $\mathcal{C}$ *be a packing in the plane with circular disks of radius at most* 1. *Consider the set of disks* $\mathcal{C}'$ *in which each disk* $C \in \mathcal{C}$ *is enlarged by a factor of* 1.00001 *from its center. Then* $\mathcal{C}'$ *covers no square with side length* 4.

From Lemma 4 and Theorem 5 we deduce our main result:

Theorem 6 *The minimum geometric dilation* Δ *necessary to embed any finite set of points in the plane satisfies* $\Delta \geq (1 + 10^{-11})\pi/2$.

Proof. (Sketch) Consider the set $P := \{\,(x,y) \mid x,y \in \{-9,-8,\ldots,9\}\,\}$ of grid points with integer coordinates in the square $Q_1 := [-9,9]^2 \subset \mathbb{R}^2$, see Figure 3. We use a proof by contradiction and assume that there exists a planar connected graph G that contains P (as vertices or on its edges) and satisfies $\delta(G) \leq (1+10^{-11})\pi/2 < 2$. In the full paper we

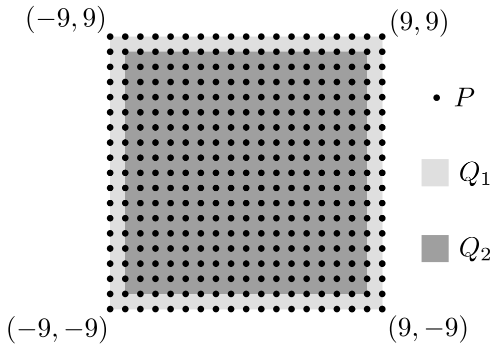

Figure 3: The point set $P := \{-9,-8,\ldots,9\}^2$ and the squares $Q_1 := [-9,9]^2$ and $Q_2 := [-8,8]^2$

show that if G attains such a low dilation, G contains a collection $\mathcal{M}$ of cycles with disjoint interiors which cover the smaller square $Q_2 := [-8,8]^2$. The length of each cycle $C \in \mathcal{M}$ is bounded by 8π implying that every disk encircled by C has a radius $r \leq 4$. Additionally, the dilation of every $C \in \mathcal{M}$ is at most $\delta(G) \leq (1 + 10^{-11})\pi/2$. Hence, Lemma 4 shows that every C has to be contained in an 1.00001-ring. It follows that the inner disks of these rings are disjoint and their 1.00001-enlargements cover Q_2 in contradiction to Theorem 5 (situation scaled by 4).

We would like to use the cycles bounding the faces of G for $\mathcal{M}$. Indeed, $\delta(G) < 2$ implies that they cover Q_2 (analogous to Figure 4b). However, their dilation could be bigger than the dilation $\delta(G)$ of the

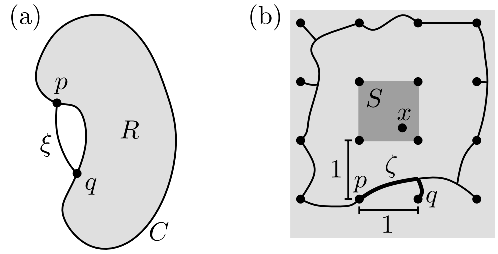

Figure 4: (a) The path ξ is a shortcut for some points of C. (b) Every $x \in Q_2$ is encircled by a cycle of length $\leq 12 \cdot \delta(G)$.

graph, see Figure 4a. G can offer shortcuts in the exterior of C, i.e., the shortest path between $p, q \in C$ does not necessarily use C.

Therefore, we have to find a different class of disjoint cycles covering Q_2 which do not allow shortcuts. The idea is to consider for every point x in Q_2 the shortest cycle of G such that x is contained in the open region bounded by the cycle. The regions of these cycles cannot intersect partly, we have $R_1 \cap R_2 = \emptyset$ or $R_1 \subseteq R_2$ or $R_2 \subseteq R_1$. If we define $\mathcal{M}$ to contain only the cycles maximal with respect to inclusion of their regions, it provides all the properties we need. Due to space limitations we can not prove all of them here.

However, one argument is displayed in Figure 4b. Every $x \in Q_2$ is contained in a square S of the integer grid. A shortest path ζ of G connecting neighbor points p, q of P next to S cannot enter S because $|\zeta| \leq \delta(G)|pq| < 2$. Hence, the concatenation of 12 such shortest paths contains a cycle of length $\leq 12\delta(G) \leq 12(1 + 10^{-11})\pi/2 \leq 8\pi$ encircling x. This shows that the regions of $\mathcal{M}$ cover Q_2 and that the length of every $C \in \mathcal{M}$ is bounded by $|C| < 8\pi$. $\square$

4 Conclusion

Our result looks like a very minor improvement over the easier bound $\Delta \geq \pi/2$, but it settles the question whether $\Delta > \pi/2$ and has required the introduction of new techniques. Our approximations are not very far from optimal, and we believe that new ideas are required to improve the lower bound to, say, $\pi/2 + 0.01$. An improvement of the constant 1.00001 in the disk packing result of [11] (Theorem 5) would of course immediately imply a better bound for the dilation.

We do not know whether the link between disk packing and dilation that we have established works in the opposite direction as well: Can one construct a graph of small dilation from a "good" circle packing (whose enlargement by a "small" factor covers a large area)? If this were true (in some meaningful sense which would have to be made precise) it would mean that a substantial improvement of the lower bound on dilation cannot be obtained without proving, at the same time, a strengthening of Theorem 5 with a larger constant than 1.00001.

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