On Geometric Properties of Enumerations of Axis-Parallel Rectangles [1]

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Abstract

We show that for any set of non-overlapping axis-parallel rectangles in the plane, there exists a *sloping* enumeration, such that the numbers of rectangles intersected by any line with a non-negative slope increase along this line. Such enumeration can be computed in the optimal time $\Theta(n \log n)$ using linear space. The notion of a sloping enumeration can be generalized to higher dimensions; however, already in three-dimensional space it may not exist. We also consider a strip packing problem for a set of rectangles with a fixed enumeration, which is required to be sloping for the resulting packing. This problem is proved to be NP-hard in any dimension $d \geq 2$.

Key words: rectangles, enumeration, orthogonal packing, NP-hardness

1. Introduction

A *Young diagram* is a collection of boxes, arranged in left-justified rows, with a (weakly) decrasing number of boxes in each row (Fig. 1a). A *standard Young tableau* is obtained by placing the numbers 1, 2, . . . , n in the n boxes of the diagram in such way that the numbers increase across each row and down each colomn (Fig. 1b).

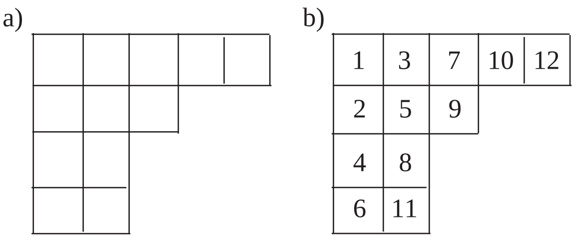

Fig. 1. a) A Young diagram; b) a standard Young tableau.

Young diagrams were introduced by Alfred Young in 1900 as a combinatorial tool (see [9]); at present combinatorics of Young tableaux has a wide range of applications in algebraic geometry and representation theory (see for example [6]).

Sometimes Young diagrams are written upside down (Fig. 2a).

Instead of considering a Young tableau as a combinatorial structure, let us look at its geometric representation: consider a standard Young tableau written upside down as a set of enumerated unit squares drawn in the plane, touching their boundaries. It is easy to see that for any line l with a non-negative slope, the numbers of squares intersected by l increase along l (Fig. 2a).

We generalize this observation in the following way: for any set of non-overlapping axis-parallel rectangles in the plane, there exists a *sloping* enumeration, such that for any line l with a non-negative slope, the numbers of rectangles intersected by l increase along l (Fig. 2b). Such enumeration can be efficiently computed in $\Theta(n \log n)$ time and $O(n)$ space. These results can be viewed as a new interpretation of our recent results [3].

All standard Young tableaux corresponding to a given Young diagram can be obtained as a topological ordering of vertices of a particular graph associated with the diagram (see [9]). All sloping enumerations for a set R of rectangles can be ob-

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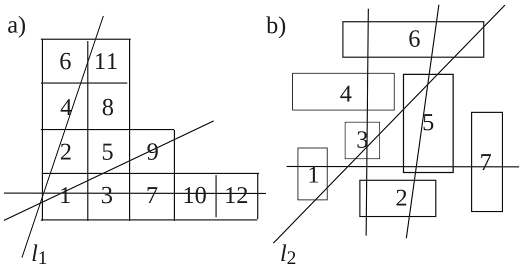

Fig. 2. a) A Young tableau written upside down; line l_1 consequently intersects squares 1, 2, 4, 6, 11. b) An enumerated set of rectangles; line l_2 consequently intersects rectangles 1, 3, 5, 6. For both a) and b), numbers of squares/rectangles intersected by any line with a non-negative slope increase along this line.

tained in the same way from a *placement graph* associated with R.

It is straightforward to generalize the notion of a sloping enumeration to higher dimensions; however, we provide an example showing that even in three-dimensional space, a sloping enumeration may not exist.

A *strip packing problem* is a special type of orthogonal packing problems (see the classification schemes introduced in [5,11]); much research has been recently carried out on problems of this kind (see the references in [5]). We consider the strip packing problem in the following form: given a set of enumerated rectangles and a horizontal strip of height H, we have to pack all rectangles into the strip, in such way that the given enumeration would be sloping for the resulting packing. As a matter of fact, we usually follow these restrictions when packing our luggage – since, for example, we do not want heavy objects to press down fragile ones. Asking for the minimal length of the strip, sufficient to store all rectangles, is shown to be NP-hard. This result can be generalized to an arbitrary dimension.

2. Preliminaries

Further we shall assume that all rectangles are axis-parallel, and no two of them overlap. We borrow terminology from [1,2,4].

For a rectangle p, let us denote its lower left, upper left, upper right and lower right vertices by A_p, B_p, C_p, and D_p, respectively. A *zone* $Z(p)$ of rectangle p is an open lower left quadrant built from point C_p (Fig. 3a). Given a set R of rectangles, its *placement graph* G_R has a vertex for each rectangle $p \in R$; for two vertices p and q, G_R contains an arc (p, q) iff $p \cap Z(q) \neq \emptyset$ (Fig. 3b).

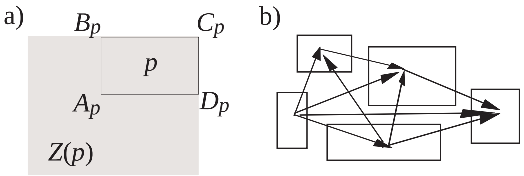

Fig. 3. a) Rectangle p and its zone $Z(p)$; b) a set of rectangles and the corresponding placement graph.

The following properties were observed in [2,10]:

Lemma 1 *Let $p, q \in R$. If $p \cap Z(q) \neq \emptyset$, then $q \cap Z(p) = \emptyset$.*

Theorem 2 *G_R is acyclic.*

3. Sloping Enumerations

Theorem 2 implies that the vertices of G_R can be topologically sorted; it follows (see [2,10,1]) that:

Theorem 3 *For any set R, there exists an enumeration $I_R : R \leftrightarrow \{1, 2, \ldots, N\}$, where $N = |R|$, such that $\forall p \in R$: $\cup\{Z(q)|I_R(q) < I_R(p)\} \cap p = \emptyset$.*

It is easy to prove that enumeration I_P satisfies our requirements.

Theorem 4 *I_R is sloping.*

PROOF. Consider a line l with a non-negative slope; let us assume that l intersects at least two rectangles. Denote the intersected rectangles by $p_{k_1}, \ldots, p_{k_m}$, according to the order, in which they are intersected by l (Fig. 4).

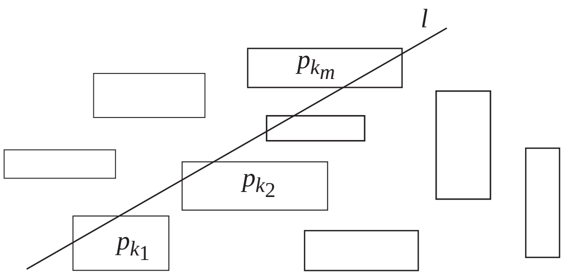

Fig. 4. Line l consequently intersects rectangles p_{k_1}, p_{k_2}, $\ldots$, p_{k_m}. For $1 \leq i < j \leq m$, we have $p_{k_i} \cap Z(p_{k_j}) \neq \emptyset$.

Obviously, $1 \leq i < j \leq m$ implies $p_{k_i} \cap Z(p_{k_j}) \neq \emptyset$, and thus, $I_R(p_{k_i}) < I_R(p_{k_j})$ must hold. It can be easily proved by induction that $I_R(p_{k_1}) < \cdots < I_R(p_{k_h})$ for any $1 \leq h \leq m$; this completes our proof.

A sloping enumeration need not be unique. To build *some* sloping enumeration, one may construct the placement graph G_R and sort topologically its vertices.

Graph G_R can be constructed with a plane sweep algorithm in $O(n \log n + |E_R|)$ time and $O(n)$ space, where $|E_R|$ is the number of its edges. This algorithm is optimal in the comparison tree model. The sorting can then be performed in $O(n + |E_R|)$ time; thus, computing a sloping enumeration in this way would take us $O(n \log n + |E_R|)$ time.

The method described above is not optimal; however, a sloping enumeration can be obtained in the optimal $\Theta(n \log n)$ time and in linear space using a plane sweep algorithm; the reader is referred to [3] for details.

4. Strip Packing Problem

Let us consider a two-dimensional strip packing problem with respect to a given enumeration (SPPE-2).

The optimization problem is:

SPPE-2: Given a set of enumerated rectangles and a horizontal strip of height H, we ask for the minimal length L of the strip, sufficient to pack all rectangles in such way that the given enumeration is sloping for the resulting packing.

The corresponding decision problem is:

SPPE-2*: For a given a set of enumerated rectangles and a horizontal strip of height H and length L, is there a feasible packing, for which the given enumeration is sloping?

Theorem 5 *SPPE-2 is NP-hard.*

PROOF. Clearly, SPPE-2*∈NP. Let us reduce to SPPE-2* the PARTITION problem, which is known to be NP-complete [8,7]. In PARTITION, we are given a finite set A and a size $s(A) \in Z^+$ for each $a \in A$, and we ask if there exists a subset $A' \subseteq A$, such that $\sum_{a \in A'} s(a) = \sum_{a \in A \setminus A'} s(a)$.

For set $A = \{a_i\}_{i=1}^n$, let $H = 5$, and $L = 2\sum_{i=1}^n s(a_i)$. For each $a_i \in A$, we construct rectangle $r(a_i)$ of size $2s(a_i) \times 2$. In addition, we construct two rectangles r_0 and r_{n+1}, each of size $\sum_{i=1}^n s(a_i) \times 2$. We enumerate rectangles as follows: $I_R(r_0) = 1$, $I_R(r_{n+1}) = n + 2$, $I_R(r(a_i)) = i + 1$, for $1 \leq i \leq n$.

Obviously, the length of the strip needed to pack all rectangles is at least L. It is precisely L iff the answer for the PARTITION decision problem is "yes".

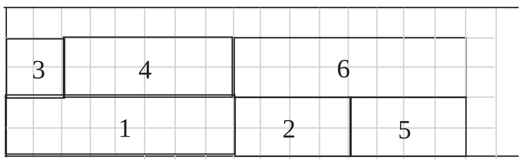

Fig. 5.

Fig. 5 illustrates the reduction described above for set $A = \{a_1, a_2, a_3, a_4\}$, with sizes $s(a_1) = 2$, $s(a_2) = 1$, $s(a_3) = 3$, $s(a_4) = 2$. Thus, $L = 16$; $H = 5$. Choosing $A' = \{a_1, a_4\}$ gives us a partition, and the packing shown on Fig. 5 fits into a strip of size $H \times L$. Clearly, our enumeration is sloping for this packing.

5. Higher-Dimensional Case

The notion of a sloping enumeration can be easily generalized to higher dimensions. In d-dimensional space, we shall consider d-dimensional *boxes* instead of rectangles.

Let us consider a set of vectors $V = \{\mathbf{v} | \mathbf{v} = \sum_{i=1}^d a_i \mathbf{e_i}, a_i \geq 0, 1 \leq i \leq d\}$, where $\mathbf{e_i}$ are unit vectors pointing along coordinate axes. Enumeration I_R of a set R of boxes is *sloping*, if for any line l, such that $\exists \mathbf{v} \in V : l || \mathbf{v}$, the numbers of boxes intersected by l increase along l.

For $d = 2$, this definition is equivalent to the one given before.

In the three-dimensional case, a sloping enumeration may not exist. A construction from [2], shown on Fig. 6, illustrates such situation. Notice that if we set in the definition $a_1 \leq 0$, $a_2, a_3 \geq 0$, there *will be* a sloping enumeration. However, in [3] we introduce a construction, for which no sloping enumeration exists for any choice of signs for a_1, a_2,

a_3. (In [3], it serves as an example of a set of parallelepipeds admitting no right-angled cut partition.)

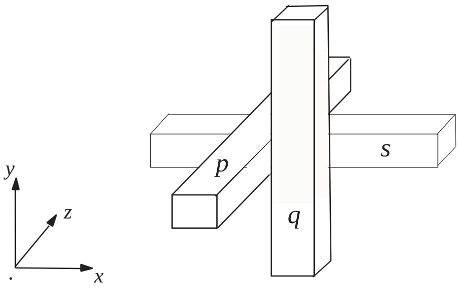

Fig. 6. For the set $\{p, q, s\}$, there exists no sloping enumeration. However, there will be a sloping enumeration, if we set $a_1 \leq 0$, $a_2, a_3 \geq 0$: $I_R(q) = 1$, $I_R(s) = 2$, $I_R(p) = 3$.

Let us generalize to higher dimensions the strip packing problem.

SPPE-*d*: Given a set of enumerated boxes in d-dimensional space, we ask for the minimal length W_1 of a container, sufficient to pack all boxes in such way that the given enumeration is sloping for the resulting packing, where the sizes in the other $d-1$ dimensions $W_2, \ldots, W_d$ are fixed.

SPPE-d*: For a given a set of enumerated boxes in d-dimensional space and a container of size W^d, is there a feasible packing, for which the given enumeration is sloping?

Theorem 6 *SPPE-d is NP-hard.*

To prove this theorem, we apply reduction similar to the one described above, setting $W_2 = 5$, $W_3 = \cdots = W_d = 1$, and requiring box $r(a_i)$ to have size $2s(a_i) \times 2 \times 1 \cdots \times 1$, $1 \leq i \leq n$, and boxes r_0 and r_{n+1} – to have size $\sum_{i=1}^{n} s(a_i) \times 2 \times 1 \cdots \times 1$.

6. Conclusion

For an arbitrary set of non-overlapping axis-parallel rectangles in the plane, we have proved existence of a sloping enumeration, and proposed methods for computing it efficiently. We have also considered a strip packing problem for an enumerated set of rectangles; an additional requirement concerned with the enumeration is the following: the given enumeration must be sloping for the resulting packing. This problem is shown to be NP-hard.

We have generalized the notion of a sloping enumeration to the case of d-dimensional space. For the three-dimensional case, we gave an example of a set of parallelepipeds, which admits no sloping enumeration. However, the strip packing problem may be stated for any dimension d; we have proved it to be NP-hard in arbitrary dimension.

All valid enumerations arise as a topological ordering of vertices of a placement graph. This graph is known to be acyclic [2,10]; however, we suppose that much more could be derived on its structure. We are going to consider this question, along with the opposite one: what kind of dags can appear as placement graphs?

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