On the Number of Pseudo-Triangulations of Certain Point Sets [1]

Oswin Aichholzer [a,2], David Orden [*,b,3], Francisco Santos [c,3] and Bettina Speckmann [d]

[a] *Institute for Software Technology, Graz University of Technology, Austria.*
[b] *Mathematics Department, University of Alcalá, Spain.*
[c] *Department of Mathematics, Statistics and Computer Science, University of Cantabria, Spain*
[d] *Department of Mathematics and Computer Science, Eindhoven University of Technology*

Abstract

We compute the exact number of pseudo-triangulations for two prominent point sets, namely the so-called double circle and the double chain. We also derive a new asymptotic lower bound for the maximal number of pseudo-triangulations which lies significantly above the related bound for triangulations.

Key words: Counting, Pseudo-triangulations, Triangulations, Double circle, Double chain.

1. Introduction

Pseudo-triangulations, a.k.a. geodesic triangulations, generalize triangulations and have found multiple applications in Computational Geometry in the last years. They were originally studied in the context of visibility [10,11] and ray shooting [5,6], but have been used in kinetic collision detection [1,8], rigidity [16], and guarding [15].

A *pseudo-triangle* is a polygon with exactly three vertices, called *corners*, with internal angles less than π. A *pseudo-triangulation* of a set S of points in the plane is a partition of the convex hull of S into pseudo-triangles whose vertex set is exactly S. A vertex is called *pointed* if it has an adjacent angle greater than π. *Pointed pseudo-triangulations*, are the ones with all vertices pointed.

The set of all pseudo-triangulations of a point set has somewhat nicer properties than that of all triangulations. For example, pseudo-triangulations of a point set with n elements form the vertex set of a certain polyhedron of dimension $3n-3$ [9]. The diameter of the graph of pseudo-triangulations is $O(n \log n)$ [3] versus the $\Theta(n^2)$ diameter of the graph of triangulations of certain point sets. For standard triangulations it is not know which sets of points have the fewest or the most triangulations, but it was shown in [2] that sets of points in convex position minimize the number of pointed pseudo-triangulations.

Let A be a point set and let A_I be its subset of interior points. The pseudo-triangulations of A can naturally be stratified into 2^{A_I} sets. More precisely, for each subset $W \subseteq A_I$ we denote by $PT_W(A)$ the set of pseudo-triangulations of A in which the points of W are pointed and those of $A_I \setminus W$ are non-pointed. For example, $PT_\emptyset(A)$ is the set of triangulations of A and $PT_{A_I}(A)$ is the set of pointed pseudo-triangulations of A. The following conjecture is implicit in previous work:

Conjecture 1 *For every point set A in general position in the plane, the cardinalities of $PT_W(A)$ are monotone with respect to W. That is to say, for any $W \subseteq A_I$ and for every $v \in W$, one has*

$$|PT_W(A)| \geq |PT_{W\setminus\{v\}}(A)|.$$

Note that [12] proves the following inequality in the other direction:

$$3\,|PT_{W\setminus\{v\}}(A)| \geq |PT_W(A)|.$$

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* Corresponding author
Email addresses: oaich@ist.tugraz.at (Oswin Aichholzer), david.orden@uah.es (David Orden), santosf@unican.es (Francisco Santos), speckman@win.tue.nl (Bettina Speckmann).
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In this paper we compute the number of pseudo-triangulations, and so check Conjecture 1, for two prominent point sets, namely those with the asymptotically maximal and minimal number of triangulations known so far:

1.1. *Points in almost convex position.*

For any given pair of numbers (v, i), $3 \le i \le v$, a point set in almost convex position with parameters (v, i) consists of the v vertices of a convex v-gon and a set of i interior points, placed sufficiently close to i different edges of the v-gon.

This is a special case of the "almost-convex polygons" studied in [7]. There it is shown that the number of triangulations of such a point set does not depend on the choice of the i edges of the v-gon. Indeed, if we call this number $n(v, i)$ one has

$$n(v, i) = n(v + 1, i - 1) - n(v, i - 1)$$

and from this $n(v, i)$ can be computed recursively, starting with $n(v, 0) = C_{v-2}$ (the Catalan number). The array obtained by this recursion (difference array of Catalan numbers) appears in Sloane's Online Encyclopedia of Integer Sequences [14] with ID number A059346 (note that there $n(v, i)$ appears for every $i \ge 0$ and $v \ge 2$, although only the cases $v \ge \max\{i, 3\}$ have an interpretation as counting triangulations). Asymptotically, $n(v, i)$ equals $4^v 3^i$, modulo a polynomial factor.

The extremal case with $v = i = n/2$ (referred to as *double circle*), has asymptotically $\Theta(\sqrt{12}^n n^{-3/2})$ triangulations. It is conjectured in [4] that this is the smallest number of triangulations that a point set with n points can have.

1.2. *Double chain.*

For any two numbers $l, m \ge 0$, a double chain is a convex 4-gon with l and m points, respectively, placed forming concave chains next to opposite edges of the 4-gon in a way that the they do not cross the two diagonals of the convex 4-gon (see Fig. 1). The double chain decomposes into a convex $l+2$-gon, a convex $m+2$-gon, and a non-convex $l+m+4$-gon, which have C_l, C_m, and $\binom{l+m+2}{l+1}$ triangulations respectively. Hence, the double chain has

$$C_l C_m \binom{l+m+2}{l+1}$$

triangulations. In the extremal case $l = m = (n - 4)/2$ this gives $\Theta(8^n n^{-7/2})$. This is (asymptotically) the point set with the largest number of triangulations known so far.

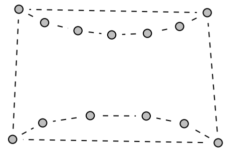

Fig. 1. A double chain: $l = 5$ and $m = 4$.

2. The double circle and its relatives

Fix two integers i, v, $3 \le i \le v$, and let A be a point set in almost convex position with parameters (v, i). Let p be a specific interior point of A and let qr be the convex hull edge which has p next to it. Let B and C be the point sets obtained respectively by deleting p from A and by moving p to convex position across the convex hull edge qr:

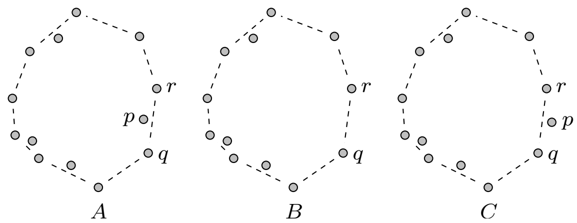

Fig. 2. The point sets A, B and C. Here $v = 9$ and $i = 4$.

Lemma 2 *Let W be a set of interior points of A not containing p (so that W is also a set of interior points of B and C). Then:*

(i) $|PT_W(A)| = |PT_W(C)| - |PT_W(B)|$.

(ii) $|PT_{W \cup \{p\}}(A)| = 2\,|PT_W(C)| - |PT_W(B)|$.

Corollary 3 *(i) A satisfies Conjecture 1.*

(ii) The numbers $|PT_W(A)|$ depend only on v, i and $k := |W|$.

We omit the proofs of these and other results due to the limited space available for this abstract.

Let $n(v, i, k)$ denote the numbers referred to in part 2 of the corollary. Since $n(v, i, 0) = 4^v 3^i$ (modulo a polynomial factor) and since

$$n(v, i, k) = 2(v+1, i-1, k-1) - n(v, i-1, k-1),$$

we conclude that $n(v, i, k) \sim 4^v 3^{i-k} 7^k$, modulo a polynomial factor. Adding the numbers over all the possible subsets of interior points gives

$$\sum_{k=0}^{i} \binom{i}{k} 4^v 3^{i-k} 7^k = 4^v 10^i.$$

Hence, a double circle (the case $i = v = n/2$) has $\sqrt{28}^n$ pointed pseudo-triangulations and $\sqrt{40}^n$ pseudo-triangulations in total, modulo a polynomial factor.

3. The single chain

As a step towards the study of the double chain, let us start with a single chain. By this we mean a point set A with three extremal vertices and a concave chain of l points next to an edge. Equivalently, a convex $l+2$-gon together with a point in its exterior and which sees all but one of its edges. We call this special point the *top* point and denote it by p. Let $p_0, \ldots, p_{l+1}$ be the rest of the points, numbered from left to right, so that the interior points are $p_1, \ldots, p_l$. We define $A_I = \{p_1, \ldots, p_l\}$.

We are particularly interested in the pointed pseudo-triangulations of the single chain. We classify them according to which interior points are joined to the top. For any subset $W \subset A_I$ we denote by $PPT_W(A)$ the set of pointed pseudo-triangulations of A in which p is joined to p_i if and only if $p_i \in W$. Clearly, $PPT_\emptyset(A)$ is in bijection to the set of triangulations of the convex $l+2$-gon, hence its cardinality is the Catalan number C_l.

Lemma 4 *For every W:*

$$|PT_W(A)| = \sum_{W' \subset W} |PPT_{W'}(A)|.$$

Hence, Conjecture 1 holds for A.

As a special case of this lemma, $|PT_\emptyset(A)| = |PPT_\emptyset(A)|$. That is to say, triangulations of A are in bijection to triangulations of the convex $l+2$-gon. Curiously enough, $PPT_{A_I}(A)$ (that is, the pointed pseudo-triangulations in which the top point p is joined to everything), have the cardinality of the next Catalan number C_{l+1}, and flips between them form the graph of the corresponding associahedron (see [13], Section 5.3 and the remark and picture on pp. 728–729). The following is a 1-dimensional analog of Conjecture 1.

Conjecture 5 *For every $W \subset A_I$ and $p \in A_I \backslash W$,*

$$|PPT_{W\cup\{v\}}(A)| \geq |PPT_W(A)|.$$

Unfortunately, we do not know how to compute the numbers $PPT_W(A)$, or even recursive formulae for them. But we can compute the sum of all the $PPT_W(A)$'s for each cardinality of W.

Theorem 6 *Let $a(l,i) := \sum_{|W|=i} |PPT_W(A)|$.*

(i) $a(l,0) = C_l$, and $a(l,1) = (l+1)C_l$.

(ii) For every $i \geq 2$,

$$a(l,i) = \binom{l+1}{i} C_l - a(l-1, i-2).$$

Part 2 of Theorem 6 allows to compute all the values of $a(l,i)$ recursively, starting from those stated in part 1. The following table shows the first few values:

$l \setminus i$	0	1	2	3	4	5	$\sum_{i=0}^{l} a(l,i)$
0	1						1
1	1	2					3
2	2	6	5				13
3	5	20	28	14			67
4	14	70	135	120	42		381
5	42	252	616	770	495	132	2307

The recursion also tells us that the array $a(l,i)$ coincides with the sequence with ID A062991 in [14]. There, we learn that the row sums, that is, the numbers $|PT_{A_I}(A)|$ of pointed pseudo-triangulations of these point sets, form the sequence A062992 and satisfy:

$$|PT_{A_I}(A)| = 2\sum_{j=0}^{l} (-1)^{l-j} C_j 2^j - (-1)^l.$$

Corollary 7 *The following inequalities hold for the number of pointed pseudo-triangulations of a single chain of l interior points, $l \geq 2$:*

$$2^l C_l < 2^{l+1} C_l - 2^l C_{l-1} < |PT_{A_I}(A)| < 2^{l+1} C_l.$$

In particular, the number is in $\Theta(8^l l^{-3/2})$.

4. The double chain

Let A be a double chain with l and m interior points in the two chains, resp. (so A has $l+m+4$ points in total). We call the $l+2$ and $m+2$ vertices in the two chains the "top" and "bottom" parts. We show how to count the number of pointed pseudo-triangulations of A.

Let us call B and C single chains with l and m interior points each. B can be considered the subset of A consisting of the top part plus a bottom vertex, and analogously for C. Every pseudo-triangulation T_A of A induces pseudo-triangulations T_B and T_C of B and C as follows: consider on the one hand all the pseudo-triangles of T_A that use at most one vertex of the bottom, and contract these vertices to a single one. Do the same for pseudo-triangles with at most one vertex in the top (see Fig. 3).

Conversely, given a pair of pseudo-triangulations of B and C, if i (resp. j) denotes the number of interior edges incident to the bottom (resp. top) point, there are exactly $\binom{i+j+2}{i+1}$ ways to recover a pseudo-triangulation of A from that data, by shuf-

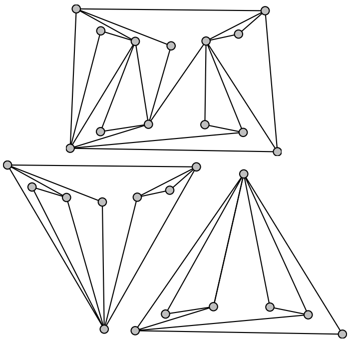

Fig. 3. Decomposing double chain pseudo-triangulations.

fling the $i+1$ pseudo-triangles of T_B incident to the bottom and the $j+1$ of T_C incident to the top.

Theorem 8 *Let V and W be subsets of the top and bottom interior points. For each $V' \subset V$ and $W' \subset W$ let $t^{V,W}_{V',W'} = \binom{l-|V\setminus V'|+m-|W\setminus W'|+2}{l-|V\setminus V'|+1}$. Then:*

$$|PT_{V\cup W}(A)| = \sum_{\substack{V'\subset V\\ W'\subset W}} t^{V,W}_{V',W'} |PPT_{V'}(B)||PPT_{W'}(C)|.$$

Corollary 9 *If Conjecture 5 holds, then the double chain satisfies Conjecture 1.*

For pointed pseudo-triangulations of the double chain, Theorem 8 says that:

$$|PT_{A_I}(A)| = \sum_{i=0}^{l}\sum_{j=0}^{m} \binom{i+j+2}{i+1} a(l,i)a(m,j),$$

where $a(\cdot,\cdot)$ is as in the previous section. The sequence for $l=m$ is

$$2, 38, 1476, 81310, 5495276, 424398044, \ldots$$

In order to analyze the asymptotics of this sequence we need the following lemma on the numbers $a(l,i)$:

Lemma 10

$$1 - \frac{i(i-1)}{(4l-2)(l-i+2)} \leq \frac{a(l,i)}{\binom{l+1}{i}C_l} \leq 1$$

Corollary 11 *Let A be a double chain with n points and with equal numbers on both sides (that is to say, $l = m = (n-4)/2$). Then:*

$$\Omega(12^n n^{-9/2}) \leq |PT_{A_I}(A)| \leq \mathrm{O}(12^n n^{-3/2}).$$

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