A Completion of Hypotheses Method for 3D-Geometry. 3D-Extensions of Ceva and Menelaus Theorems [1]

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Abstract

A method that automates hypotheses completion in 3D-Geometry is presented. It consists of three processes: defining the geometric objects in the configuration; determining the hypothesis conditions of the configuration (through a point-on-object declaration method); and applying an algebraic automatic theorem proving method to obtain and prove the sufficiency of complementary hypothesis conditions. To avoid as much as possible the appearance of rational expressions, projective coordinates are used (although affine and Euclidean problems can also be treated). A Maple implementation of the method has been used to extend to 3D classic 2D geometric theorems like Ceva's and Menelaus'.

Key words: 3D-Geometry, Simbolyc Computation, Automatic Theorem Proving

1. Brief Description of the Method

Hypotheses completion was already treated by Recio and Vélez [6]. The method presented in this paper automates hypotheses completion in 3D-Geometry. Let us give a brief description of its three processes.

1.1. *Defining the Geometric Objects in the Configuration*

Among the geometric objects in a configuration, some can be defined directly and others are determined through geometric operations (see Table 1). Other usual geometric objects included in the package (segment, midpoint, sphere, quadric,...) are omitted for the sake of space.

The desired configuration can be constructed through the adequate concatenation of these *elementary* commands. Note that in this Geometry not only the rule-and-compass *globally constructible* objects can be treated: those geometric objects such that any of their points can be constructed with rule-and-compass, can be treated too.

Projective coordinates are used. Command `intCoor` allows to substitute coordinates where rational expressions appear by the corresponding integer quaternions.

1.2. *Determining the hypothesis conditions of the configuration*

Hypothesis conditions are declared as membership relations between points and higher dimension geometric objects. To declare $P = [p_0, p_1, p_2, p_3]$ as a point on the object ϕ (being the equations of ϕ: $\phi_i(x_0, x_1, x_2, x_3) = 0$; $i = 1, ..., n$) is equivalent to impose that the *hypothesis conditions* $\phi_i(P_0, P_1, P_2, P_3) = 0$; $i = 1, ..., n$ are verified. Command `pointOnObject` takes care of adding these polynomials to a certain list, denoted $LREL$, where the *hypothesis polynomials* are stored, and to add the corresponding variables to the list VAR.

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Object	Input	Command	Output
initial point (free point)	four projective coordinates	`point`	list of 4 parameters
plane	three non-collinear points	`plane`	equation of the plane
line	two different points	`line`	list of equations of the line
point on line AB ($\overrightarrow{PB} = r \cdot \overrightarrow{PA}$)	two points (A, B) and a real number r	`rateOnLine`	list of coords. of point P
plane/line parallel to a given plane/line	one linear object and one point	`parallel`	equation(s) of the plane/line
plane/line perpendicular to a given line/plane	one linear object and one point	`perpendicular`	equation(s) of the plane/line
intersection of two objects (not necessarily linear)	two already defined objects	`intersection`	coords. of point(s) or equation(s) of linear objects or reduced list of eqs. (in GB sense)

Table 1
Geometric objects' definition

1.3. *Obtaining and Proving the Sufficiency of Complementary Hypothesis Conditions*

In most configuration geometric problems, the thesis is (or can be reduced to) a $P \in \phi$ membership condition (where P is a point and ϕ is a geometric object) or to a geometric relation among geometric objects in the configuration. In both cases the *thesis polynomial* admits a $\phi(P)$ form.

In case list $LREL$ is empty, to check that the thesis holds is equivalent to check that ϕ vanishes in P (i.e., that $\phi(P) = 0$). Command `isPlaced` applied to the pair (P, ϕ) takes care of performing all the corresponding computations.

In case list $LREL$ is not empty, to check that the thesis holds it is sufficient to check that ϕ can be expressed as an algebraic linear combination of the polynomials in list $LREL$, what can be effectively computed using Wu's techniques. A brief description of these automatic proving techniques can be found in [1], meanwhile a detailed description can be found, e.g., in [2,9]. These techniques were adapted to hypotheses completion in [5] and to geometric loci determining in [7]. The technique described in this paper is essentially that of [7], but has been adapted to the way hypothesis and thesis conditions are usually declared.

This process basically consists of two steps:

- to triangularize system $LREL$ w.r.t. the variables in list VAR, to obtain system $TRIP$
- to compute, starting with $\phi(P)$, the successive pseudo-remainders of dividing by the polynomials in $TRIP$ w.r.t. the variables in VAR, until the last pseudo-remainder (polynomial ω) is obtained.

That $\omega = 0$ is a *necessary condition* for the thesis to hold. Command `newHypot` of our package, applied to (P, ϕ), automatically computes ω.

But we would still have to check that $\omega = 0$ is a *sufficient condition* for the thesis $\phi(P) = 0$ to hold.

If a parametrization of $\omega = 0$ can be obtained, then we substitute in $\phi(x_0, x_1, x_2, x_3)$ the x_i by their corresponding parametric expressions. If the resulting polynomial vanishes, then condition $\omega = 0$ is also sufficient. Command `isPlaced` can take care of these computations.

If a parametrization of $\omega = 0$ can't be obtained, then ω is be added to list $LREL$, and the new variable appearing in ω but not in list VAR, is added to list VAR. The same process can be applied now, and, if the last pseudo-remainder is 0, then condition $\omega = 0$ is also sufficient. Command `autProve` can take care of these computations.

2. 3D-Extension of Ceva and Menelaus Theorems

An application of the automatic theorem proving method described above is included as illustration afterwards. The goal is to determine conditions that make four points, lying on consecutive edge-lines of a tetrahedron, coplanary (see Figure 1). This problem was recently solved using synthetic techniques by H. Davis [3].

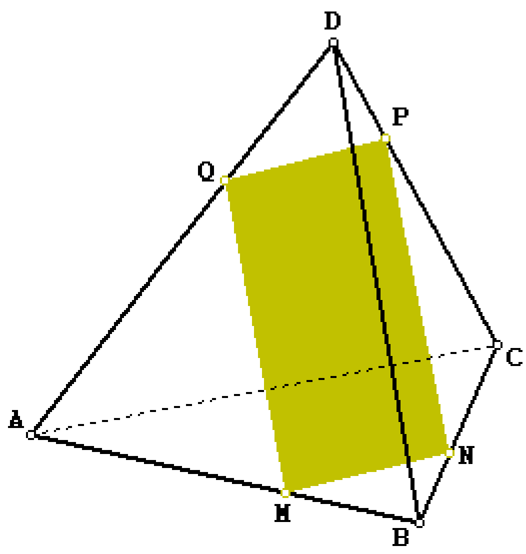

Fig. 1. Extending to 3D Ceva and Menelaus theorems

We can assume that the vertices are $A(1,0,0,0)$, $B(1,1,0,0)$, $C(1,\gamma_1,\gamma_2,0)$, $D(1,\delta_1,\delta_2,\delta_3)$ without any lack of generality (these points can be defined using command `point`). Given $m,n,p,q \in \mathbb{R}\cup\{\infty\}$, let M,N,P,Q be the points lying on the edge-lines AB, BC, CD, DA (respectively), and satisfying

$$\overrightarrow{MB} = m \cdot \overrightarrow{MA} \quad ; \quad \overrightarrow{NC} = n \cdot \overrightarrow{NB}$$
$$\overrightarrow{PD} = p \cdot \overrightarrow{PC} \quad ; \quad \overrightarrow{QA} = q \cdot \overrightarrow{QD}$$

(they can be defined using command `rateOnLine`). Then plane MNP can be defined (using command `plane`).

As detailed above, applying command `newHypot` to the pair (Q, MNP), a necessary condition for Q to lie on plane MNP (i.e., for M,N,P,Q to be coplanary): $-\gamma_2 \cdot \delta_3 \cdot (-1 + m \cdot n \cdot p \cdot q) = 0$, is obtained. As A,B,C,D are non-coplanary points, and consequently, $\gamma_2 \neq 0 \neq \delta_3$, what implies: $m \cdot n \cdot p \cdot q = 1$. To verify that is a sufficient condition, Q is particularized for $q = 1/(m \cdot n \cdot p)$, and applying command `isPlaced` to the pair (Q, MNP), 0 is obtained, what confirms that Q belongs to plane MNP. This leads to the following:

Theorem 1 *Points M,N,P,Q, lying on the oriented consecutive edge-lines AB, BC, CD, DA of tetrahedron $ABCD$ (respectively), are coplanary, if and only if:*

$$(MB/MA)\cdot(NC/NB)\cdot(PD/PC)\cdot(QA/QD) = 1$$

Observe that the points M,N,P,Q do lie on the consecutive oriented edge-lines AB, BC, CD, DA, but they can lie outside the edge-segments, and therefore this result doesn't only generalizes Ceva theorem, but also Menelaus theorem.

3. Comparison with Other Methods

As the automatic theorem proving technique used in this work is based on Wu's algorithm, it is of a lower computational complexity than those techniques based on the use of Groebner bases.

Comparing this method with others based on Wu's techniques, the main difference is the way the geometric objects of the configuration are defined and the way the hypotheses conditions are declared. In the method presented here the geometric objects and the hypotheses conditions are obtained in a natural way, following the geometric algorithm that generates the configuration, instead of translating into algebraic expressions the geometric relations that determine them (what is usually the case).

That happens, for instance, in Simson-Steiner-Guzmán theorem 3D-extension [4]. The goal is to determine the conditions so that the projections (in prefixed directions) of a point on the faces of a tetrahedron are coplanary. This problem was developed in [7], translating into algebraic expressions the geometric relations. Now it has been developed using the method detailed in section 1, in a more comfortable and faster way.

Other advantage of the method proposed in Section 1 is the simple way in which parameters and variables are distinguished (what is not straightforward in other approaches). With this method the parameters are the non-numeric coordinates of the initial points (that are preserved along all subsequent calculations), meanwhile the variables are the coordinates of the point-on-object objects defined using `pointOnObject` command.

Another advantage of the method proposed in Section 1 is the possibility to develop the geometric algorithm of the configuration using a Dynamic Geometry System, and to translate it to a Computer Algebra System syntax (interpreting it using the package considered here), as already done in 2D [8]. We plan to implement it in the near future.

4. Conclusions

The hypotheses completion in 3D-Geometry method described is convenient and efficient. It allows the user to obtain automatically the equations in the configuration, the hypothesis conditions obtained directly in the configuration and the complementary hypothesis conditions that have to be added for the thesis condition to hold.

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