ON FENCING PROBLEMS

C. Miori [a] C. Peri [b] S. Segura Gomis [c]

[a] *Departamento de Análisis Matemático, Universidad de Alicante, Campus de San Vicente del Raspeig, E-03080-Alicante, Spain; borsista dell'Università Cattolica S. C. di Brescia, Italy*

[b] *Universià Cattolica S. C., Largo Gemelli 1, I-20123 Milano,Italy*

[c] *Departamento de Análisis Matemático, Universidad de Alicante, Campus de San Vicente del Raspeig, E-03080-Alicante, Spain*

Abstract

Fencing problems deal with the bisection of a convex body in a way that some geometric measures are optimized. We study bisections of planar bounded convex sets by straight line cuts and also bisections by hyperplane cuts for convex bodies in higher dimensions.

Key words: fencing problems, optimization, relative diameter.

1. Introduction

Fencing problems regard the division of a given set into two subsets of equal area in a way that some geometric measure is maximized or minimized.

Last year we pointed out results on centrally symmetric convex sets in two dimension. Now we present a statement in the general planar case, where K is a general convex body. Moreover we study fencing problems in higher dimension.

First we need to recall definitions and some results.

Definition *Let K be a planar, bounded, convex set and let $K_1, K_2 \subset K$. We say that $\{K_1, K_2\}$ is a **bisection** of K if the following conditions hold:*

(i) $K = K_1 \cup K_2$.

(ii) $int(K_1) \cap int(K_2) = \emptyset$.

(iii) $V(K_1) = V(K_2) = \frac{1}{2}V(K)$, *where $V(.)$ is the area functional.*

(iv) K_1 and K_2 are connected subsets such that $\gamma = \partial K_1 \cap \partial K_2$ is an arc of continuous curve joining two points in ∂K.

*We say that γ **bisects** K.*

Email addresses: cm4@alu.ua.es (C. Miori), carla.peri@unicatt.it (C. Peri), Salvador.Segura@ua.es (S. Segura Gomis).

In the planar case we replace the volume functional $V(.)$ by the area functional $A(.)$.

We define the *relative diameter* $d(K_1; K)$ as follows:

$$d(K_1; K) = max\{D(K_1), D(K \setminus K_1)\},$$

where $D(\cdot)$ is the diameter functional, and $\{K_1, K_2\}$ is a bisection of K.

The following propositions give us properties regarding subdivisions 1) by straight lines, 2) by general curves.

Theorem 1 *Let $\{K_1, K_2\}$ be a bisection of a planar, centrally symmetric, and bounded convex set K by a straight line l. Then*

(i) One of the ends of the diameter of K_1 is one of the intersection points of l with ∂K. Denote this point by M.

(ii) The other end of the diameter of K_1 is the intersection of ∂K with the smallest circle centered at M and containing K_1; the radius of this circle is obviously $d(K_1; K)$.

Theorem 2 *Let K be a planar, centrally symmetric, and bounded convex set. Let γ be a continuous curve bisecting K into two connected subsets E_1 and E_2. Suppose that $d(E_1; K)$ realizes the minimum of the relative diameter for all the curves bisecting K. Then there exist a straight line, passing through the* *center of K, bisecting K into two subsets $\{F_1, F_2\}$ such that $d(F_1; K) = d(E_1; K)$.*

Moreover we obtained a global estimate for the ratio between the area and the relative diameter:

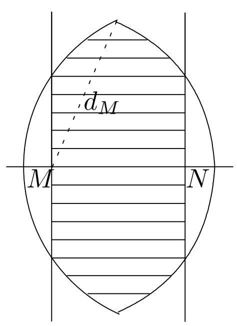
Theorem 3 *Let K be a planar, bounded, and centrally-symmetric convex set with area A. For every bisection $\{K_1, K_2\}$ of K the relative diameter satisfies the following inequality*

$$d(K_1; K) \geq C\sqrt{A}$$

where

$$C \cong 0.8815....$$

Equality holds if K is the body K_{φ_0} described in the figure:



2. Minimizing the relative diameter for planar convex sets

Now we shall consider bisections of a general convex set by straight line cuts. We prove that centrally symmetric convex sets minimize the relative diameter.

First of all we need to introduce the following notations. Let us consider a general convex set K, and let l be the straight line bisecting K; let l^+ and l^- be the half-planes defined by such a line. Let us define:

$$d(K) = min_{l \cap K \neq \emptyset} d_l(K),$$

where $d_l(K) = max\{D(K \cap l^+), D(K \cap l^-)\}$.

Theorem 4 *In the class of convex sets K with area A the minimum of the relative diameter, with respect to straight line cuts, is attained on a centrally symmetric convex set.*

PROOF. Let the area A of K be fixed, and let l_0 be the line such that

$$d_{l_0}(K) = d(K).$$

Let us choose the origin to be the midpoint of the chord $K \cap l_0$. Let us apply Steiner symmetrization with respect to $l_0^{\perp}$. Then we obtain a new body K' such that $A(K' \cap l^+) = A(K' \cap l^-) = \frac{1}{2}A$ and

$$D(K' \cap l_0^+) \leq D(K \cap l_0^+),$$

$$D(K' \cap l_0^-) \leq D(K \cap l_0^-),$$

so that

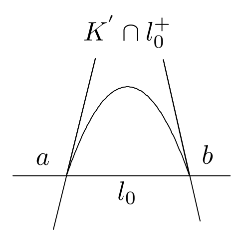
$$d_{l_0}(K') = max\{D(K' \cap l_0^+), D(K' \cap l_0^-)\} \leq$$

$$\leq max\{D(K \cap l_0^+), D(K \cap l_0^-)\} = d_{l_0}(K).$$

In order to minimize the relative diameter for fixed area it is enough to consider bodies which are symmetric with respect to $l_0^{\perp}$, where l_0 realize the minimal cut. Let K' be such a body.

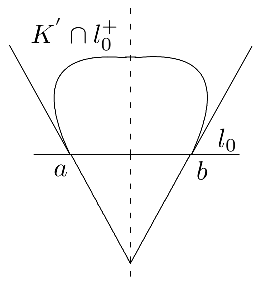
Let now $K' \cap l_0^+$ the part of K' such that $d(K') = D(K' \cap l_0^+)$. We consider two cases.

1) First case: the support lines of K' at a, b (endpoints of $K' \cap l_0$) meet at a point $p \in l^+$ or are parallel. Then we have the following situation.



Applying the symmetrization with respect to the line l_0 we obtain a new body which is centrally symmetric and has the same relative diameter as K'. We apply then the argument for centrally symmetric bodies in order to get the best constant.

2) Second case: let us suppose that the support lines of K' at a, and b meet at a point $p \in l^-$.



Since

$$d(K') = D(K' \cap l_0^+) \geq D(K' \cap l_0^-)$$

then

$$\forall x \in K' \cap l_0^+ : dist(a, x) \leq d(K')$$
$$\forall x \in K' \cap l_0^- : dist(b, x) \leq d(K')$$

Thus K' is contained in the intersection of the two disks with centers a, b respectively and radius $d(K')$. Moreover $K' \cap l_0^- \in T$ where T is the intersection of the previous lens with the triangle in the half-plane l_0^- determined by the support lines passing trough a and b. So if we take as new body $\widetilde{K}$ given by the union of T and its symmetric with respect to l_0 we have a centrally symmetric convex body where l_0 realizes a bisection having $d(\widetilde{K}) = d(K')$ and area of $\widetilde{K}$ greater then the area of K'.

3. Fencing problems on higher dimension

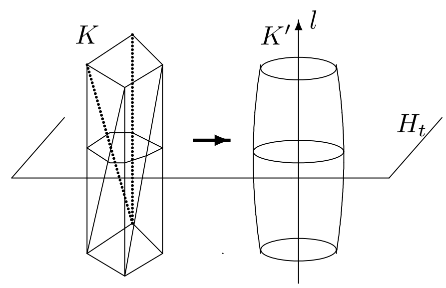
We now consider fencing problems on higher dimension determined by hyperplane cuts. We say that a convex body K on $\mathbb{E}^d$ is a *minimizer* for such a fencing problem if the relative diameter of a bisection $\{K_1, K_2\}$ of K attain the minimum value.

We give a necessary condition for a convex body K to be the minimizer. We also describe geometric properties of best bisections of centrally symmetric convex bodies and obtain a lower bound for the ratio

$$\frac{d(K_1; K)}{V(K)^{1/d}}.$$

Theorem 5 *In the class of convex bodies in $\mathbb{E}^d$ with volume V the minimum of $d(K_1; K)$, with respect to hyperplane cuts, is attained on a centrally symmetric convex body of revolution.*

PROOF. Let $K \subset \mathbb{E}^d$ be a body for which $d(K_1; K)$ attains its minimum, and let H be the hyperplane cutting K into two parts K_1, K_2 such that $d(K_1; K) = D(K_1)$. By applying Schwarz rounding symmetrization (see [2] and [12]) with respect to a line l perpendicular to H, we obtain a body of revolution K', with same volume V, which is bisected by H into two parts K_1', K_2' such that $V(K_1') = V(K_2') = V/2$. As the Schwarz symmetrization does not increase the diameter, the value of $d(K_1; K)$ does not increase, so that by the minimality of K, the body K' also realizes the minimum of $d(K_1; K)$.



Let us consider the tangent cone Γ of K' along the $(d-1)$-sphere $K' \cap H$. We distinguish two cases.

(i) Suppose that Γ is a right cylinder and denote by K_2'' the image of K_1' by the orthogonal reflection with respect to the hyperplane H. Then the convex body $K'' = K_2'' \cup K_1'$ is a centrally symmetric convex body of revolution with a bisection $\{K_2'', K_1'\}$ for which $d(K_1; K)$ attains its minimum.

(ii) Suppose that Γ is a cone with apex $v \in l$. Denote by H^+ the half-space bounded by H which contains v. Let $K_1'' = K' \cap H^+$ and let K_2'' denote the image of K_1'' by the orthogonal reflection with respect to the hyperplane H. Then the convex body $K'' = K_2'' \cup K_1''$ is a centrally symmetric convex body of revolution with a bisection $\left\{K_2'', K_1''\right\}$ for which $d(K_1; K) = D(K_2'') = D(K_1'') = D(K' \cap H^+) \leq D(K_1)$. Thus $d(K_1; K)$ attains its minimum on K'' as well.

Therefore, in order to minimize the ratio

$$\frac{d(K_1; K)}{V(K)^{1/d}}$$

it is enough to find for fixed $d(K_1; K)$ the body with maximum volume within the class of centrally symmetric bodies of revolution. To this end we first extend Proposition 1 on the planar centrally symmetric case.

Theorem 6 *Let $\{K_1, K_2\}$ be a bisection of a centrally symmetric convex body K by a hyperplane H, and let $D(K_1) = \max\{D(K_1), D(K_2)\}$. Then*
- *one of the end points of the diameter of K_1 belongs to $H \cap \partial K$. Let M be such a point.*
- *The other endpoint of the diameter is the intersection of ∂K with the sphere centered at M with radius $D(K_1)$.*

PROOF. Let A, B denote the endpoints of the diameter of K_1. We can assume that the center of K is the origin O. Suppose by contradiction that $A, B \notin H \cap \partial K$. By construction the points A, B, O are not collinear. Let us consider the plane π passing through A, B, O. We define $K' = K \cap \pi$, $K'_1 = K_1 \cap \pi$, $K'_2 = K_2 \cap \pi$. We obtain a bisection of the centrally symmetric planar convex body K' such that $\max\{D(K'_1), D(K'_2)\}$ is attained on the segment with endpoints A, B. This contradicts Proposition 1 on the planar centrally symmetric case.

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