Improved results for the k-Centrum straight-line location problem

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Abstract

The k-Centrum problem consists in finding a point that minimises the sum of the distances to the k farthest points out of a set of given points. It encloses as particular cases to two of the most known problems in Location Analysis: the center, also named as the minimum enclosing circle, and the median. In this paper the k-Centrum criteria is applied to obtaining a straight line-shaped facility. A reduced finite dominant set is determined and an algorithm with lower complexity than the previous one obtained.

Key words: Location, Duality

1. Introduction

Given a set of n demand points, the median straight line problem consists in finding the straight line that minimises the sum of the distances to the points. The center straight line problem looks for the straight line that minimises the distance to the farthest point. A k-Centrum straight line minimises the sum of the distances to the k farthest demand points. If $k = 1$ then the corresponding problem is that of the center and for $k = n$ the median, thus enclosing both problems. However, last is not the only reason for the relevancy of this problem: it provides the decision maker a more flexible tool.

The k-Centrum criteria has been applied in point location facility in different contexts. When the underlying structure is a graph, Tamir [9] has devised efficient algorithms for different cases (chain graph, tree, general graphs, multiple case). The corresponding repulsive problem, that of maximising the sum of the distances to the k nearest points has also been studied both in graphs [5] and in the Euclidean plane [4]. Some of the classic location problems have been extended to extensive facilities (i.e. those that can not be represented as isolated points). In particular, the center and the median line location problem have been intensively researched (e.g. Lee and Wu [3], Morris and Norback [7], Schöebel [8].) The only work on the k-Centrum straight line problem is [6] in which a finite dominating set is determined and an $O(n^4 \log n)$ time algorithm described.

2. Finite Dominating Set

Given a set of points $P = \{p_1, p_2, \ldots, p_n\} \subset \mathbb{R}^2$ and the weight set associated $W = \{w_1, w_2, \ldots, w_n\}$ the straight line k-Centrum problem consists in finding a line such that it minimises the function:

$$f(l) = \max_{Q \subset P, |Q|=k} \sum_{p \in Q} w_p d(p, l)$$

where $d(\cdot, \cdot)$ is the Euclidean point-line distance.

The problem can be stated in the dual plane as finding the point l^* such that it minimises the dual objective function:

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$$f^*(l^*) = \max_{Q^* \subset P^*, |Q^*|=k} \sum_{p^* \in Q^*} \frac{w_p d_v(l^*, p^*)}{\sqrt{1 + {l_x^*}^2}}$$

where $d_v(\cdot, \cdot)$ is the vertical point-line distance and l_x^* is the x-coordinate of the point l^*.

In the dual plane let us consider the **V**ertical **D**istance **C**ompletely **O**rdered **L**ine **V**oronoi **D**iagram: **VDCOLVD**(P^*). The set

$$B_v(P^*) = \{\mathrm{bis}_v(p_i^*, p_j^*); 1 \leq i \leq n, 1 \leq j \leq n, i \neq j\}$$

induces that diagram. The Voronoi regions are the union of (convex) polygonal regions neither necessarily bounded nor simply connected.

Lemma 1 *The* **VDCOLVD***(P^*) induces a partition in the set of non-vertical straight lines of the Euclidean plane such that the collection of ordered distances to the points of P remains constant in each class.*

PROOF. For each region R of the Voronoi diagram **VDCOLVD**(P^*) there exists a permutation σ of the set $\{1, 2, \cdots, n\}$ such that

$$w_{\sigma(1)} d_v(l^*, p_{\sigma(1)}^*) \leq \cdots \leq w_{\sigma(n)} d_v(l^*, p_{\sigma(n)}^*)$$

Dividing by $\sqrt{1 + {l_x^*}^2}$ it follows

$$w_{\sigma(1)} d(p_{\sigma(1)}, l) \leq \cdots \leq w_{\sigma(n)} d(p_{\sigma(n)}, l)$$

Lemma 2 *The objective function in the dual plane f^* is quasiconcave in each connected component of each Voronoi region.*

PROOF. In each such a region CR the order of distances from lines p_i^* to points in it, remains constant, i.e. $\forall\, l^* \in CR$

$$w_{\sigma(1)} d_v(l^*, p_{\sigma(1)}^*) \leq \cdots \leq w_{\sigma(n)} d_v(l^*, p_{\sigma(n)})$$

Thus the k largest weighted vertical distances remains constant in CR and, therefore, the function

$$f^*(l^*) = \frac{\sum_{i=n-k+1}^{n} d_v(l^*, p_{\sigma(i)}^*)}{\sqrt{1 + {l_x^*}^2}}$$

is the ratio between a sum of linear functions and a positive convex function, therefore becoming a quasiconcave function.

Let us note that each extreme point of the Voronoi regions could be either

(i) a vertex of a bisector of two elements of P^*,

(ii) an intersection of two edges corresponding to the bisectors of two pairs of P^* with a common element,

(iii) an intersection of two edges corresponding to the bisectors of two disjoined pairs of elements.

However, not all of these extreme points are images of candidates for the problem in the primal space.

Theorem 3 *A finite dominant set for the straight line k-Centrum problem is composed by the elements of the set of straight lines passing through two points of P and those of the set of straight lines at equal weighted distances from three points of P.*

PROOF. Candidate straight lines correspond under the geometrical dual map to candidate points for the dual objective function. Points (i) are images of straight lines passing through two points. Points (ii) correspond to straight lines at equal distances from three points.

However, points (iii) cannot be optimal points. In effect,let l^* be such a point; then there exists two pairs $\{p_i^*, p_j^*\}$ and $\{p_s^*, p_t^*\}$ such that l^* is the intersection of the corresponding bisectors. Therefore, $w_i d_v(l^*, p_i^*) = w_j d_v(l^*, p_j^*)$ and $w_s d_v(l^*, p_s^*) = w_t d_v(l^*, p_t^*)$. Ruling out the degenerate case in which coincide the four weighted distances, we may assume that $w_i d_v(l^*, p_i^*) < w_s d_v(l^*, p_s^*)$. Then, the only relevant case to be consider is when one of the two weighted distances, say $w_i d_v(l^*, p_i^*)$, is k-rank at l^*, but in this case in each halfplane above and below the $w_i d_v(l^*, p_i^*) = w_j d_v(l^*, p_j^*)$ line and within a small enough neighbourhood of l^* the set of k-closest points does not change, so the formula for f^* remains exactly the same in each of these half-neighbourhoods of l^*, so is quasiconcave in this half-neighbourhood, in which l^* is not a vertex, so cannot be optimal.

Finally, unbounded extreme points of regions in the dual space correspond to vertical lines in the primal. A small rotation argument shows that vertical straight lines not holding one of the two above conditions cannot be optimal.

3. Algorithm

In order to examine the candidates we will consider each pair $\{p_i, p_j\}$ of points and the bisector $\text{bis}_v(p_i^*, p_j^*)$ of their dual points. For the sake of simplicity the reasoning will be done on one of the two lines of the bisector. If C_{ij}^* is the set of candidate points on this line, then C_{ij} is a family of straight lines with a common point $O_{ij} = (\text{bis}_v(p_i^*, p_j^*))^*$. In order to compute the objective function, the outer hull of the given points will be used. With this purpose the problem will be transformed in a equivalent non weighted one, with given point set

$$P' = \{p'_t = O_{ij} + w_t + \overrightarrow{O_{ij}p_t} / p_t \in P\}.$$

Let us note that under this transformation the weighted distance from a point p of P to a straight line r of C_{ij} is equal to the unweighted distance from p' to r. Furthermore, for each straight line $r \in C_{ij}$ their k farthest points in P' can be found in the k outer convex hulls of P'.

A short description of the algorithm follows:

– Step 1: Compute the best vertical line.
– Step 2: For each pair $\{p_i, p_j\} \in P$,
  · Obtain the sets C_{ij} and P'.
  · Compute the k outer convex hulls for P'.
  · For each $r \in C_{ij}$ compute its objective value by obtaining their k farthest point.
– Step 3: Select the best candidate from Steps 1 and 2.

Let us note that by means of this algorithm the i-centrum ($1 \leq i \leq k$) straight lines can be computed in $\mathrm{O}(n^3 \log n)$ time, when k is fixed.

4. Prospects

The finite dominant set given in Section 2 could be too large when it is desired to find the k-Centrum straight line only for a fixed k. In this case an alternative set would result from considering the k-1 level of a plane arrangement. Each weighted straight line of the dual plane could be substituted by two planes such that the computation of the k farthest straight lines to a point is equivalent to finding the k highest planes to a point. Then by using the results in [2] and [1] the time complexity of the finite dominant set we expect would be $O(n \log n + n^{1+\epsilon})$.

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