Computing the convex hull of disks using only their chirotope

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Abstract

We show that the convex hull of a collection of n pairwise disjoint disks in the plane is computable in $O(n \log n)$ time using only the chirotope of the collection of disks. The method relies mainly on the development of an (elementary) theory of convexity in the universal covering space of the punctured plane.

1. Introduction

1.1. *Result of the paper.*

Throughout the paper we consider a finite family o_i of $n \geq 2$ pairwise disjoint bounded closed 2-dimensional convex sets in the plane with regular boundaries (*disks* for short) and we assume that the disks are in general position in the sense that there is no line tangent to three disks. A bitangent is a closed line-segment tangent to two disks at its endpoints. The chirotope of the family of disks is defined as the map χ that associates with each ordered triplet (u, v, w) of bitangents tangent to a same disk o the value $+1$ if walking counterclockwise around the boundary of o we encounter the bitangents u, v, w in cyclic order $u, v, w, u, v, \ldots$; the value -1, otherwise. The main result of the paper is the following.

Theorem 1 *The convex hull of a collection of n pairwise disjoint disks in the plane is computable in $O(n \log n)$ time using only the chirotope of the collection of disks.* $\square$

1.2. *Previous work.*

Several algorithms have been developed in the past to address the convex hull problem for disks in the plane : the set of hull-bitangents can be computed as the set of breakpoints of the upper envelope of the support functions of the o_i using a divide-and-conquer algorithm [6, chap. 6] as described in [5], running in $O(n \log n)$ time but making use of the chirotope of the family of directions of the set of bitangents of the set of disks augmented with a point at infinity. More sophisticated techniques – using even more involved predicates like slicing the disks – have been developpped to design output sensitive convex hull algorithm [4]. (The related problem of computing the convex hull of a simple curved polygon is adressed in [1].)

For point obstacles the situation is different : Graham's scan [2] and Knuth's incremental algorithm [3] both compute the convex hull of a set of points using only its chirotope.

1.3. *Motivations.*

Applied motivations We have devised an algorithm that computes a pseudo-triangulation given the convex-hull. This algorithm also runs in $O(n \log n)$ and uses only the chirotope. Now, pseudotriangulations are useful data structures.

Implementation motivations The chirotope shows fewer degenerate cases than the more involved predicates used in previous algorithms, and our algorithms use simpler data-structures.

Theoretical motivations The convex hull depends only on the chirotope, not on the more involved predicates used in existing algorithms. Computing these objects by means of the most basic predicates possible is interesting in its own right. It can also lead to isolating a family of axioms satisfied by the chirotopes, in the style of Knuth's CC-systems.

2. Convexity in the universal covering space of the punctured plane.

2.1. *Definitions and notations*

Let $p : \tilde{\mathbb{P}} \longrightarrow \mathbb{P}$ be a universal covering of the punctured plane $\mathbb{P} = \mathbb{R}^2 \setminus \{(0,0)\}$. The reader might have in mind as a model of the map p the vertical projection upon plane $z = 0$ of the screw surface $\{(r\cos\theta, r\sin\theta, \theta) \in \mathbb{R}^3 \mid (\theta, r) \in \mathbb{R} \times \mathbb{R}^{+*}\}$. θ is called the *angle-coordinate* of point (θ, r) of $\tilde{\mathbb{P}}$ and r is called its distance-coordinate. As the reader might suspect we use the geometric structure on $\tilde{\mathbb{P}}$ inherited via the covering map from the euclidean structure on the punctured plane. Thus a line (half-line, line segment) of $\tilde{\mathbb{P}}$ is a connected component of the pre-image under the covering map of a line (half-line, line-segment) of $\mathbb{P}$. A line will be regarded as oriented towards growing angle-coordinate. Define the *angle* (denoted $\hat{L}$) of a line L, as the least upper bound of the angle coordinate of its points. A line L is uniquely identified by $\hat{L}$ and the distance to the origin of its projection onto $\mathbb{P}$, denoted $\delta(L)$.

Points A and B in $\tilde{\mathbb{P}}$ are said visible if there is a line-segment in $\tilde{\mathbb{P}}$ with endpoints A and B. A subset X of $\tilde{\mathbb{P}}$ is called convex if for any pair (A, B) of visible points in X, line-segment $[AB]$ is included in X. An intersection of convex sets is a convex set, which allows to define the convex hull of a set as the smallest (for the inclusion relation) convex set containing the set. An half-plane of $\tilde{\mathbb{P}}$ is the closure of a connected component of the complement of a line of $\tilde{\mathbb{P}}$. The projection of an half-plane under p is either the whole punctured plane $\mathbb{P}$ or an half-plane of $\mathbb{P}$.

A *s*imple convex of $\tilde{\mathbb{P}}$ is a connected component of the pre-image under p of a closed bounded convex subset of $\mathbb{P}$. A *l*ong convex of $\tilde{\mathbb{P}}$ is a closed convex that contains a point (θ, r) for any θ and some r depending on θ, and that is bounded in the r direction for any θ. The pre-image under p of a closed bounded convex subset of $\mathbb{R}^2$ containing the origin of the plane in its interior is an example of a long convex.

A positive supporting line of convex is a directed line containing boundary points such that the convex lies on the left side of the directed line.

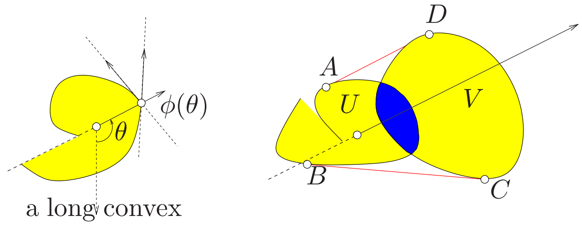

Fig. 1. A simple comvex and a long convex share 2 common exterior tangents. The drawaing is done in the sheet of the disk = the sheet of the point of the disk that realizes its distance to the origin.

2.2. *Some theorems*

Theorem 2 *The border of a long convex is a curve $\theta \rightarrow (\theta, \phi(\theta))$ that has a left-hand and a right-hand tangent for every θ, such that if they differ, the right-hand one has a larger angle than the left-hand one, and the angle of the left- or right-hand tangent increases with θ. The tangents to the border are positive supporting lines (and vice-versa).* □

Let U be a convex and let x be a point. A boundary point y of U is said visible from x if line segment $[x, y)$ exists and U are disjoint. The set of boundary points of U visible from y is a closed connected set.

Theorem 3 *Let U be a long convex, and V a simple convex, such that $V \cap U = \emptyset$ or V is not included in U and ∂U intersects ∂V in exactly two points that are not angular points of both ∂U and ∂V. Then U and V share exactly two positive common supporting lines, and the border of their convex hull is made of two half-unbounded connected sub-arcs $-\infty B$ and $A{+}\infty$ of ∂U, a connected sub-arc CD of ∂V and two bitangent segments BC and CA to U and V where BA (rep. DC) is the set of boundary points of U (resp. V) visible from V (resp. U).* □

3. Our algorithm.

3.1. *Notations.*

Wlog, assume $(0,0) \in o_1$. Every o_i $(i > 1)$ has infinitely many simple convex lift-up in $\tilde{\mathbb{P}}$, while o_1 has a single lift-up, $\tilde{o}_1$, a long convex. Choose one lift-up of o_2 and denote it $o_{2,1}$. If I and J are lift-up of o_i and o_j for some i and j, we define $v_{\epsilon I \epsilon' J}$ as the lift-up of $v_{\epsilon i \epsilon' j}$ whose endpoints belong in I and J, when such a lift-up actually exists (for instance this is always the case between $\tilde{o}_1$ and any

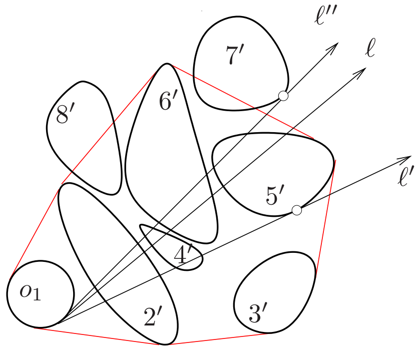

Fig. 2. In this example the disks $o_{i'}$ is labelled i', $B(\ell) = v_{1'2'}v_{2'3'}v_{3'5'}v_{5'6'}v_{6'2'}v_{2'1'}$, $D(\ell) = \{1', 2', 3', 4', 5', 6'\}$, and $Q(\ell) = [1, 2', 6', 5']$.

lift-up of any o_i with $i > 1$). Wlog, assume $v_{\tilde{o}_1 o_{2,1}}$ has angle 0. Denote $o_{i,k}$ the lift-up of o_i such that $v_{\tilde{o}_1 o_{i,k}}$ has angle in $[2(k-1)\pi, 2k\pi)$. Finally, denote $\ell_{i,k}$ the half-line, supporting $v_{\tilde{o}_1 o_{i,k}}$, with origin its tangency point upon $\tilde{o}_1$.

We perform a rotational sweep, with a half-line ℓ tangent to $\tilde{o}_1$ at its origin. The sweep starts at position $\ell = \ell_{1,1}$. During the sweep, we keep track of a subset of the objects that intersect ℓ, therefore, we will stop when ℓ reaches every $\ell_{i,k}$ when $i > 1$ (an "enter event"), and some $\ell_{i,k}$ when $i < -1$ (a "leave event"). We define a total order on the set of half-lines leaving $\tilde{o}_1$ by setting $\ell \prec \ell'$ if and only if $\hat{\ell} < \hat{\ell}'$. Computationally, if $i > 0$ and $j > 0$, then $\ell_{i,k} \prec \ell_{j,l}$ if and only if $k < l$ or $k = l$ and $\chi(v_{12}, v_{1i}, v_{1j})$ (or $i = 2$), so that we can sort the events (if, say $i < 0$ and $\chi(v_{1-i}, v_{12}, v_{1i})$, substitute $k+1$ to k). Then for a position ℓ of the sweep half-line, we define :

(i) the counterclockwise sequence $B(\ell) = v_1 v_2 \ldots v_k$ of bitangents of the convex hull $C(\ell)$ of the set $D(\ell)$ of disks $o_{j,k}$ such that $\ell_{j,k} \preceq \ell$ – by convention $\tilde{o}_1$ is an element of $D(\ell)$ –, starting from $v_1 = v_{\tilde{o}_1 o_{2,1}}$; the arc with source v_i and sink v_{i+1} is denoted a_i and its supporting object o'_i (a_0 and a_k are half-unbounded loops around $\tilde{o}_1$). In fact, we sometimes regard B as the list of arcs a_i.

(ii) the arc $a(\ell)$ defined as the arc $a_{j'}$ where j' is the minimal element of the subset of indexes j ($1 \le j \le k$) such that ℓ pierces the supporting disks o'_i of a_i ($j \le i \le k$) in the order $o'_k, o'_{k-1}, \ldots, o'_j$ ($o'_k = o_1$). The list $o'_k o'_{k-1} \ldots o'_{j'}$ is denoted $Q(\ell)$.

See Figure 2 for an illustration. We write ℓ^- for any half-line of the open interval (ℓ', ℓ) where ℓ' is the previous event.

During the sweep, we maintain $B(\ell)$, $Q(\ell)$ and $a(\ell)$. Initially $\ell = \ell_{2,1}$, $D(\ell) = \{\tilde{o}_1, o_{2,1}\}$, $B(\ell) = [v_{\tilde{o}_1 o_{2,1}}; v_{o_{2,1}\tilde{o}_1}]$, $a(\ell)$ is the arc with source $v_{\tilde{o}_1 o_{2,1}}$ and sink $v_{o_{2,1}\tilde{o}_1}$, and $Q(\ell) = \tilde{o}_1 o_{2,1}$. We store Q in a binary search tree. To that end, we need to be able to decide given two objects $o_{i,k}$ and $o_{j,l}$ intersecting ℓ which one is to the right of the other along ℓ. This can be done with the procedure "if $\chi(v_{1j}, v_{1i}, v_{1-j})$ then $\chi(v_{ij}, v_{1i}, v_{i-j})$ else $\chi(v_{1j}, vji, v_{j-i})$".

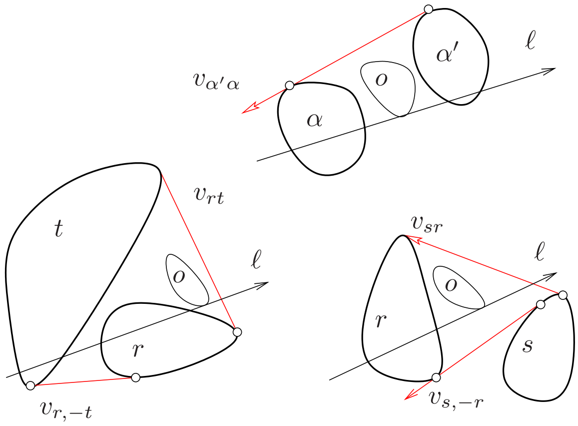

Fig. 3. o_i is included in the current convex hull.

3.2. *Handling enter-events.*

Assume we are to process the enter-event for object o. Let r be the rightmost disk of $Q(\ell^-)$, s its predecessor along $B(\ell^-)$, and t its successor (if any, i.e., if $r \neq \tilde{o}_1$).

Theorem 4 *l pierces either $a(\ell^-)$, or $v_{r(\ell^-)t(\ell^-)}$ or $v_{s(\ell^-)r(\ell^-)}$.*

Theorem 5 *Assume that o appears along ℓ at the left of disk r, between, say, disks α and α'. Then either o is included in $C(\ell^-)$, or o intersects $v_{\alpha'\alpha}$. The latter case occurs if and only if $p(o)$ intersects bitangent $v_{p(\alpha')p(\alpha)}$, and then the conditions of theorem 3 are satisfied.* □

Theorem 6 *Assume that o appears along ℓ at the right of disk r. Then either arc $a(\ell^-)$ contains a point visible from o or o is included in $C(\ell^-)$. The latter case stands if and only if either (non exclusive this time) $p(o)$ is included in the convex hull of $p(r)$ and $p(t)$ (in that case, ℓ pierces v_{sr} or $v_r t$), or $p(o)$ is included in the pseudotriangle bounded by the bitangents $v_{p(s)p(r)}$ and $v_{p(s),-p(r)}$, and $p(s)$ is not above $p(\ell)$ and $s + (2\pi, 0)$ has not been inserted yet* *(in that case l pierces through $v_{s(\ell)r(\ell)}$). See Figure 3 for an illustration. Also, when o is not included in $C(\ell^-)$, the conditions of theorem 3 are satisfied.* $\square$

Now, we explain how to update B and Q. First we locate o in Q.

Assume first that o *is to the left of* r. Let α and α' be its left-hand and right-hand neighbours in $Q(\ell^-)$. If $p(o)$ does not intersect $v_{p(\alpha)p(\alpha')}$ we just ignore $o : B(\ell) = B(\ell^-)$, $Q(\ell) = Q(\ell^-)$, and $a(\ell) = a(\ell^-)$. Otherwise we insert o into Q, and update B : we split B at v, and we regard the two resulting parts as stacks of arcs whose respective heads are the arcs contributed by α and α'. Then we pop from the left-hand stack until an arc β, say supported by disk o', is met such that $o' = \tilde{o}_1$ or $\chi(u, v, w)$ where u is the source of $p(\beta)$, w is its sink, and $v = v_{p(o)p(o')}$. Similarly we pop from the right-hand stack until an arc β', say supported by disk o'', is met such that $\chi(u, v', w)$ where u is the source of $p(\beta')$, w is its sink, and $v'' = v_{p(o'')p(o)}$. Then, we shorten β (respectively β') : its source (respectively sink) is replaced with $v_{oo'}$) (respectively $v_{o''o}$). Then, to build $B(\ell)$, we concatenate what is left of the two stacks, with the arc (denoted δ) of ∂o with source $v_{o''o}$ and sink $v_{oo'}$ in between. When an arc that follows $a(\ell^-)$ (included) in $B(\ell^-)$ is popped, its supporting disk is removed from Q. If $a(\ell^-)$ is removed from the right-hand stack, denoting u the object supporting the predecessor of δ along $B(\ell)$, we insert u into q if $u + (2\pi, 0)$ has not undergone an enter-event yet and $p(u)$ intersects $p(\ell)$ at the right of $p(o)$, so that u becomes $r(\ell)$ instead of o.

Assume now that o *is to the right of* r. We discard o when one of the two cases stated in theorem 6 holds. Otherwise, we proceed as in the previous case, with three exceptions : we split $B(\ell^-)$ through arc a instead of bitangent $v_{\alpha'\alpha}$ (that is, there is one copy of a at the head of both stacks); the copy of a at the top of the left-hand stack is popped if and only if $\chi(v_{ro}, v_{or}, v_{rt})$; an object is removed from Q if only if an arc it supports is popped from the left-hand stack.

The predicates used to decide whether o should be discarded can be implemented by means of χ, we omit the details.

3.3. *Handling leave events.*

In fact, only leave events for r need to be processed. The processing of those events simply consists in removing r from Q.

3.4. *Sketch of a proof of correction.*

From theorems 5, 6 and 3, if follows that : (1-) when o is included in $C(\ell^-)$, B is not updated, so that $B(\ell)$ is still the border of $C(\ell)$; (2-) when o is not included in $C(\ell^-)$, the border of $C(\ell)$ is made of two connected arcs of the border of $C(\ell^-)$, one connected arc of the border of o and two bitangents to o. What our algorithm does is a two-way walk along the border of $C(\ell^-)$, starting from a point known to be in the arc of $C(\ell^-)$ that is to be discarded, until discovery of the two bitangents.

3.5. *Extracting the planar convex hull.*

We could prove that after only two rounds (that is, at $\ell^-_{2,3}$), $p(B)$ contains the convex hull of $o_1, \ldots, o_n$ as a factor. But we do not know how to identify efficiently that factor based solely on the chirotope. Hence the need for a third round. Denote B the value of B obtained at the end of the third round.

Theorem 7 *Let γ be the first arc of B (for counter-clockwise orientation) supported by some $o_{i,2}$. Then γ indeed exists, the bitangent $b' = b + (2\pi, 0)$ appears in B (it is created during the third round), and the projection of the factor $\gamma M b'$ of B is the convex hull of the o_i.* $\square$

References

[1] C. Bajaj and M. S. Kim. Convex hull of objects bounded by algebraic curves. *Algorithmica*, 6:533–553, 1991.

[2] R. L. Graham. An efficient algorithm for determining the convex hull of a finite planar set. *Inform. Process. Lett.*, 1:132–133, 1972.

[3] D. E. Knuth. *Axioms and Hulls*, volume 606 of *Lecture Notes Comput. Sci.* Springer-Verlag, Heidelberg, Germany, 1992.

[4] F. Nielsen and M. Yvinec. Output-sensitive convex hull algorithms of planar convex objects. *Internat. J. Comput. Geom. Appl.*, 8(1):39–66, 1998.

[5] D. Rappaport. A convex hull algorithm for discs, and applications. *Comput. Geom. Theory Appl.*, 1(3):171–181, 1992.

[6] M. Sharir and P. K. Agarwal. *Davenport-Schinzel Sequences and Their Geometric Applications.* Cambridge University Press, New York, 1995.