Smoothed Number of Extreme Points under Uniform Noise

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Abstract

We analyze the maximal expected number of extreme points of a point set P in $\mathbb{R}^d$ that is slightly perturbed by random noise. We assume that each point in P is uniformly distributed in an axis-aligned hypercube of side length 2ϵ centered in the unit hypercube (the center of the hypercube can be regarded as the point position without noise). Our model is motivated by the fact that in many applications the input data is inherently noisy, e.g. when the data comes from physical measurement or imprecise arithmetic is used. For this input distribution we derive an upper bound of $\mathcal{O}((n \cdot \log n/\epsilon)^{1-1/(d+1)})$ on the number of extreme points of P.

Key words: Randomization, Smoothed Analysis

1. Introduction

The convex hull of a point set in the d-dimensional Euclidean space is one of the fundamental combinatorial structures in Computational Geometry. Many of its properties have been studied extensively in the last decades. In this paper we are interested in the number of vertices of the convex hull of a random point set, sometimes referred to as the *number of extreme points* of the point set. It is known since nearly 30 years that the number of extreme points of a point set drawn uniformly at random from the (unit) hypercube is $\mathcal{O}(\log^{d-1} n)$, cf. [1]. The number of extreme points has also been studied for many other input distributions, e.g. for Gaussian normal distribution. In this paper we consider the expected number of extreme points when each input point is chosen from a (possibly) different small subcube of the unit hypercube. We can think of this input distribution as resulting from some point set P (defined by the centers of the subcubes), where each point is afflicted with some small random noise. Our model is motivated by the fact that in many applications the input data is inherently noisy, e.g. when the data comes from physical measurement or imprecise arithmetic is used.

1.1. *Related Work*

Several authors have treated the structure of the convex hull of n random points. In 1963/64 Rényi and Sulanke, [8] and [9], were the first to present the mean number of extreme points in the planar case for different stochastic models. They showed for n points uniformly chosen from a convex polygone with r vertices a bound of $r \cdot \log n$. This work was continued by Efron [4], Raynaud [6] and [7], and Carnal [2] and extended to higher dimensions. For further information we refer to the excellent book [10] by Santaló.

In 1978 Bentley, Kung, Schkolnick and Thompson [1] showed that the expected number of extreme points of n i.i.d. random points in d space is $\mathcal{O}(\ln^{d-1} n)$ for fixed d for some general probability distributions. Har-Peled gave in [5] a different proof of this result. Both results are based on the computation of the expected number of maximal points (cf. Section 3).

The concept of smoothed analysis was introduced in 2001 by Spielman and Teng [11]. In 2003

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this concept was applied to the number of changes to the combinatorial describtion of the smallest enclosing bounding box of a moving point set [3] where different probability distributions for the random noise were considered. This work already covers the two dimensional case of this paper's problem. By considering moving points the two dimensional problem was reduced to a one dimensional probelm. The general extension to higher dimensions holds some non trivial difficulties even for the case when uniformle distributed noise is considered.

2. Problem Statement and Uniform Case

Let $P := \{p_1, \ldots, p_n\}$ denote a point set in the unit hypercube $[0,1]^d$ and let $\mathcal{V}(P)$ be the number of *extreme points* (i.e., the number of vertices of the convex hull) of P. Furthermore, let $r_1, \ldots, r_n$ be i.i.d. random vectors chosen uniformly at random from $[-\epsilon, \epsilon]^d$ and let $\widetilde{p_1} := p_1 + r_1, \ldots, \widetilde{p_n} := p_n + r_n$ be the perturbed points and $\widetilde{P}$ the set of perturbed points. Then we define the *smoothed number* of extreme points to be $\mathcal{V}(\widetilde{P}) := \max_P \mathbf{E}[\mathcal{V}(\widetilde{P})]$.

Our bounds are based on the following observation: A point $p \in P$ is not extreme, if each of the 2^d orthants centered at p contains at least one point. In this case, we say that p is not *maximal.* It follows immediatly that the number of maximal points is an upper bound on the number of extreme points. Therefore, we will from now on count the number of maximal points.

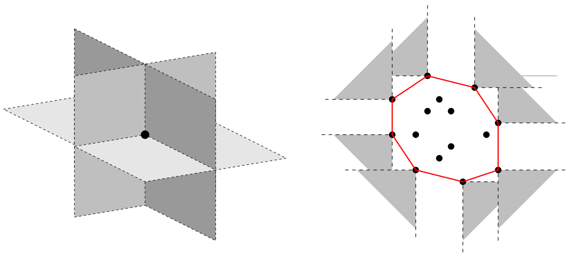

Fig. 1. 'A point in three dimensional space has eight orthants.' *and* 'Every extreme point is also a maximal point.'

As a warm-up we illustrate our approach on the well-understood case of n points chosen uniformly at random from the d-dimensional unit hypercube. We show how to obtain in this case an upper bound of $\mathcal{O}(\log^{d-1} n)$ on the number of maximal points and hence on the number of extreme points.

Theorem 1 *Let $P = \{p_1, \ldots p_n\}$ be a set of n points chosen uniformly at random from the d-dimensional unit hypercube. Then the expected number of extreme points of P is $\mathcal{O}(\log^{d-1} n)$ for fixed dimension d.*

Proof : To prove the theorem we show that

$$\mathbf{Pr}[p_i \text{ is maximal}] = \mathcal{O}(\log^{d-1} n/n) \ . \tag{1}$$

By linearity of expectation it follows immediately that the number of extreme points is $\mathcal{O}(\log^{d-1} n)$. To prove (1) we consider the probability that a fixed orthant $\o(p_i)$ centered at p_i is empty. Using a standard union bound we get

$$\mathbf{Pr}[p_i \text{ is maximal}] \leq 2^d \cdot \mathbf{Pr}[\o(p_i) \text{ is empty}] \ .$$

Wlog. we now fix orthant $\o(p_i) := \prod_{j=1}^{d} [-\infty, p_i^{(j)}]$. We can write the probability that $\o(p_i)$ is empty as an integral in the following way: consider p_i having the coordinates $(x^{(1)}, \ldots, x^{(d)})$. The probability for any other point $p_k \in P \setminus \{p_i\}$ to be not in $\o(p_i)$ is then equal to $1 - x^{(1)} \cdot x^{(2)} \cdots x^{(d)}$. Since there are $n-1$ other points in P the probability that $\o(p_i)$ is empty is exactly

$$\int_0^1 \cdots \int_0^1 (1 - x^{(1)} \cdots x^{(d)})^{n-1} \mathrm{d}x^{(1)} \cdots \mathrm{d}x^{(d)} \ . \tag{2}$$

We solve this integral by repeatetd substitution and demonstrate this on the 2 dimensional integral. We start with the integral $\int_0^1 \int_0^1 (1-xy)^{n-1} \mathrm{d}x\mathrm{d}y$ and substitute in a first step $1 - xy =: z = z(x)$ which gives us $\mathrm{d}z = -y \cdot \mathrm{d}x$ and $z(0) = 1$ and $z(1) = 1 - y$:

$$\int_0^1 \int_0^1 (1-xy)^{n-1} \mathrm{d}x\mathrm{d}y \ = \ \int_0^1 \int_{1-y}^1 \frac{z^{n-1}}{y} \mathrm{d}z\mathrm{d}y \ = \ \int_0^1 \left[\frac{1}{n} \cdot \frac{z^n}{y} \right]_{1-y}^1 \mathrm{d}y \ = \ \frac{1}{n} \int_0^1 \frac{1}{y} \cdot (1 - (\underbrace{1-y}_{=:\, z})^n) \mathrm{d}y \ .$$

Now we substitute $1 - y =: z = z(y)$ and we get $\mathrm{d}z = -1 \cdot \mathrm{d}y$ and $z(0) = 1$ and $z(1) = 0$:

$$\frac{1}{n} \int_0^1 \frac{1}{1-z} \cdot (1 - z^n) \mathrm{d}z \ = \ \frac{1}{n} \int_0^1 \sum_{i=0}^{n-1} z^i \mathrm{d}z \ = \ \frac{1}{n} \left[\sum_{i=0}^{n-1} \frac{z^{i+1}}{i+1} \right]_0^1 \ = \ \frac{1}{n} \sum_{i=1}^{n} \frac{1}{i} \ = \ \frac{\log n + \mathcal{O}(1)}{n} \ .$$

Since there are 4 quadrants and n points it follows that the expected number of maximal points in the planar case is $\mathcal{O}(\log n)$.

The d-dimensional integral (2) boils down to the sum

$$\frac{1}{n}\sum_{i_1=1}^{n}\frac{1}{i_1}\sum_{i_2=1}^{i_1}\frac{1}{i_2}\cdots\sum_{i_{d-1}=1}^{i_{d-2}}\frac{1}{i_{d-1}}=\mathcal{O}\left(\frac{\log^{d-1} n}{n}\right),$$

which proves the theorem. □

3. Smoothed Number of Extreme Points

We now want to apply the same approach to obtain an upper bound on the smoothed number of extreme points. We consider again a perturbed point $\widetilde{p_i}=p_i+r_i$, where $p_i=(p_i^{(1)},\ldots,p_i^{(d)})\in[0,1]^d$ and r_i a random vector chosen uniformly from $[-\epsilon,\epsilon]^d$. It follows that $\widetilde{p_i}$ lies in the hypercube $\prod_{j=1}^{d}[p_i^{(j)}-\epsilon,p_i^{(j)}+\epsilon]=:\mathrm{phc}(p_i)$ which we want to call the perturbation hypercube of point p_i.

Now we recall that $\o(\widetilde{p_i})=\prod_{j=1}^{d}[-\infty,\widetilde{p_i}^{(j)}]$. For any other perturbed point $\widetilde{p_k}=p_k+r_k$, $i\neq k$, the probability that $\widetilde{p_k}$ does not lie in $\o(\widetilde{p_i})$ is

$$\mathbf{Pr}[\widetilde{p_k}\notin\o(\widetilde{p_i})]=\int_{\mathcal{I}(\widetilde{p_i},p_k)}\left(\frac{1}{2\epsilon}\right)^d\mathrm{d}y$$

where $\mathcal{I}(\widetilde{p_i},p_k)$ is the set of all valid positions for $\widetilde{p_k}$ not lying in $\o(\widetilde{p_i})$, i.e. $\mathcal{I}(\widetilde{p_i},p_k)=\mathrm{phc}(p_k)-\o(\widetilde{p_i})$. Note that $1/(2\epsilon)^d$ is the probability density of $\widetilde{p_k}$. We get

$$\mathbf{Pr}[\o(\widetilde{p_i})\text{ is empty}]=$$

$$\int_{\mathrm{phc}(p_i)}\left(\frac{1}{2\epsilon}\right)^d\cdot\left(\prod_{k\neq i}\int_{\mathcal{I}(x,p_k)}\left(\frac{1}{2\epsilon}\right)^d\mathrm{d}y\right)\mathrm{d}x\ .$$

The main idea is now to subdivide the unit hypercube into $m=1/\delta^d$ smaller axis-aligned hypercubes of sidelength δ. Then we subdivide P into sets $C_1,\ldots,C_m$ where C_ℓ is the subset of P that is located (before the perturbation) in the ℓ-th small hypercube (we assume some ordering among the small hypercubes). Now we can calculate the expected number $\mathcal{D}(C_\ell)$ of maximal points for the sets C_ℓ and use

$$\mathcal{V}(\widetilde{P})\ \le\ \sum_{\ell=1}^{m}\mathcal{V}(\widetilde{C}_\ell)\ \le\ \sum_{\ell=1}^{m}\mathcal{D}(\widetilde{C}_\ell)\tag{3}$$

to obtain an upper bound on the expected number of extreme points in $\widetilde{P}$. The advantage of this approach is that for small enough δ the points in a single small hypercube behave almost as in the uniform random case.

We now want to compute the expected number of extreme points for the sets C_ℓ. We assume wlog. that C_ℓ is the hypercube $[0,\delta]^d$. Let $\bar\delta=(\delta,\ldots,\delta)$ and $\bar 0=(0,\ldots,0)$ etc. We now want to find an upper bound on the probability that $\o(\widetilde{p_i})$ is empty. This probability is maximized, if $p_i=\bar 0$ and $p_k=\bar\delta$ for every $p_k\in C_\ell$, $i\neq k$. Hence, we get

$$\mathbf{Pr}[\o(\widetilde{p_i})\text{ is empty}]\le$$

$$\int_{\mathrm{phc}(\bar 0)}\left(\frac{1}{2\epsilon}\right)^d\cdot\left(\int_{\mathcal{I}(x,\bar\delta)}\left(\frac{1}{2\epsilon}\right)^d\mathrm{d}y\right)^{n-1}\mathrm{d}x\ .$$

Using $\mathcal{I}(x,\bar\delta)=[\delta-\epsilon,\delta+\epsilon]^d-\prod_{j=1}^{d}[-\infty,x^{(j)}]=\prod_{j=1}^{d}[\max\{\delta-\epsilon,x^{(j)}\},\delta+\epsilon]$ we can write the integral as

$$\int_{[-\epsilon,\epsilon]^d}\left(\frac{1}{2\epsilon}\right)^d\cdot\Bigg(\sum_{\mathrm{I}\subseteq[d]}(-1)^{|\mathrm{I}|+1}\cdot\int_{\max\{(\delta-\epsilon)_{\mathrm{I}},x_{\mathrm{I}}\}}^{(\delta+\epsilon)_{\mathrm{I}}}\left(\frac{1}{2\epsilon}\right)^{|\mathrm{I}|}\mathrm{d}y_{\mathrm{I}}\Bigg)^{n-1}\mathrm{d}x\tag{4}$$

where the subscript I for a variable denotes the $|\mathrm{I}|$ dimensional projection of the variable to the coordinates in I, i.e. $x_{\mathrm{I}}=(x^{(j_1)},\ldots,x^{(j_t)})$ for $\mathrm{I}=\{j_1,\ldots,j_t\}$, and the maximum is taken coordinate wise. Let

$$\mathrm{F}(x,j):=\sum_{\mathrm{I}\subseteq[d-j]}(-1)^{|\mathrm{I}|+1}\int_{(\delta-\epsilon)_{\mathrm{I}}}^{(\delta+\epsilon)_{\mathrm{I}}}\left(\frac{1}{2\epsilon}\right)^{|\mathrm{I}|}\mathrm{d}y_{\mathrm{I}}+\sum_{\mathrm{I}\subseteq[j]}(-1)^{|\mathrm{I}|+1}\int_{x_{\mathrm{I}}}^{(\delta+\epsilon)_{\mathrm{I}}}\left(\frac{1}{2\epsilon}\right)^{|\mathrm{I}|}\mathrm{d}y_{\mathrm{I}}\ .$$

We can rewrite the integral (4) now in the form

$$\sum_{j=0}^{d}\binom{d}{j}\cdot\underbrace{\int_{-\epsilon}^{\delta-\epsilon}\cdots\int_{-\epsilon}^{\delta-\epsilon}}_{d-j}\underbrace{\int_{\delta-\epsilon}^{\epsilon}\cdots\int_{\delta-\epsilon}^{\epsilon}}_{j}\left(\frac{1}{2\epsilon}\right)^d\cdot\mathrm{F}(x,j)^{n-1}\mathrm{d}x\ .\tag{5}$$

Next we want to calculate this integral. Again we will use repeated substitution. We obtain the following result, which we prove only for the one dimensional case (the complete proof is deferred to the full version of this paper).

Lemma 2 *Let $C_\ell \subseteq P$ be a set of c_ℓ points in the hypercube $[0,\delta]^d$ before the perturbation takes place. The smoothed number of extreme points in $\widetilde{C}_\ell$ is at most*

$$\mathcal{V}(\widetilde{C}_\ell) \ \le\ \mathcal{D}(\widetilde{C}_\ell) \ \le\ c_\ell \cdot 2^d \cdot \left(d \cdot \frac{\delta}{2\epsilon} + \frac{\log^{d-1} c_\ell}{c_\ell} \right) \ .$$

Proof : *(only the 1-dimensional case)*

Let us consider the one dimensional integral. From (5) we have

$$\int_{-\epsilon}^{\delta-\epsilon} \frac{1}{2\epsilon} \cdot \left(\int_{\delta-\epsilon}^{\delta+\epsilon} \frac{1}{2\epsilon} \mathrm{d}y \right)^{n-1} \mathrm{d}x + \int_{\delta-\epsilon}^{\epsilon} \frac{1}{2\epsilon} \cdot \left(\int_{x}^{\delta+\epsilon} \frac{1}{2\epsilon} \mathrm{d}y \right)^{n-1} \mathrm{d}x \ =$$

$$\int_{-\epsilon}^{\delta-\epsilon} \frac{1}{2\epsilon} \mathrm{d}x \ + \int_{\delta-\epsilon}^{\epsilon} \frac{1}{2\epsilon} \cdot \left(\frac{1}{2\epsilon} \cdot \underbrace{(\delta+\epsilon-x)}_{=:\,z} \right)^{n-1} \mathrm{d}x \ .$$

The first integral solves to $\delta/(2\epsilon)$. In the second integral we substitute $\delta+\epsilon-x =: z = z(x)$ and we get $\mathrm{d}z = -1 \cdot \mathrm{d}x$ and $z(\delta-\epsilon) = 2\epsilon$ and $z(\epsilon) = \delta$. Thus it is

$$\int_{\delta-\epsilon}^{\epsilon} \frac{1}{2\epsilon} \cdot \left(\frac{1}{2\epsilon} \cdot (\delta+\epsilon-x) \right)^{n-1} \mathrm{d}x \ =$$

$$\int_{\delta}^{2\epsilon} \left(\frac{1}{2\epsilon} \right)^{n} \cdot z^{n-1} \, \mathrm{d}z \ = \ \frac{1}{n} \cdot \left(1 - \left(\frac{\delta}{2\epsilon} \right)^{n} \right) \ .$$

It follows that in the one dimensional case the probability for a point to be maximal is at most

$$2 \cdot \left(\frac{\delta}{2\epsilon} + \frac{1}{n} \right) \ .$$

□

We can now conclude from (3) and Lemma 2 that

$$\mathcal{V}(\widetilde{P}) \ \le\ \sum_{\ell=1}^{1/\delta^d} c_\ell \cdot 2^d \cdot \left(d \cdot \frac{\delta}{2\epsilon} + \frac{\log^{d-1} c_\ell}{c_\ell} \right) \ .$$

Our main theorem follows choosing

$$\delta = \mathcal{O}\left(\left(\frac{\epsilon \cdot \log^{d-1} n}{d \cdot n} \right)^{1/d} \right) \ .$$

Theorem 3 *The smoothed number of extreme points of a set $\widetilde{P}$ of n perturbed points in d dimensional space with start points from the unit hypercube and under uniform noise from $[-\epsilon,\epsilon]^d$ is*

$$\mathcal{V}(\widetilde{P}) = \mathcal{O}\left(\left(\frac{n \cdot \log n}{\epsilon} \right)^{1-\frac{1}{d+1}} \right) \ .$$

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