Unfolding Simple Chains Inside Circles

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Abstract

It is an open problem to determined whether a polygonal chain can be straightened inside a confining region if its links are not allowed to cross. In this paper we propose a special case: whether a polygonal chain can be straightened inside a circle without allowing its links to cross. We prove that this is possible if the straightened configuration can fit within circle. Then we show that these simple chains have just one equivalence class of configurations.

Key words: folding, polygonal chain, reconfiguration

1. Introduction

A *chain* is a sequence of rigid rods or links consecutively connected at their endpoints, about which they may rotate freely. The link between A_{i-1} and A_i $(1 \le i \le n)$ is denoted by L_i and the length of L_i is denoted by l_i. The angle at intermediate joint A_i, $\theta_i \in [0, 2\pi)$, is that determined by rotating L_i about A_i counterclockwise to bring L_i to L_{i+1}. The chain Γ is *simple* if it is non-self-crossing and non-self-touching. The subchain of Γ with joints $A_i, ..., A_j$ is denoted by $\Gamma[i, j]$.

We say a *bend operation* is performed at joint A_i, when the joint angle θ_i is changed between θ_i and π. Throughout this paper, we assume that the only bend operations allowed are *single-joint* bend operations, in which only one joint angle is altered at a time. A bend operation is *complete* if, at the end of the operation the joint angle is π. We then say that the joint has been *straightened.* A bend Operation that is not complete is called a *partial bend.* A sequence of bend operations is said to be *monotonic* if no operation increases the absolute deviation from straightness, $|\theta_i - \pi|$, for a joint A_i.

Let $\sigma = (i_1, i_2, ..., i_{n-1})$ be a permutation of the indices $\{1, 2, ..., n-1\}$. For a simple chain Γ, we say that a sequence $(A_{i_1}, A_{i_2}, ..., A_{i_{n-1}})$ of joints is *unfoldable*, if Γ can be straightened into a straight line segment L using the joints in the sequence in turn, such that Γ remains simple and all of the bend operations are complete. A simple chain Γ is called *unfoldable chain*, if it has a unfoldable sequence of joints. An intermediate joint A_i is called *unfoldable joint*, if a complete bend operation can be performed at A_i such that during the performing bend operation, Γ remains simple.

The *union chain*, Γ_U, of a chain Γ is a chain which is obtained from Γ in the following way: if none of the joints of Γ is straight joint, $\Gamma_U = \Gamma$; if Γ has at least one straight joint, for any straight joints A_i, we delete joint A_i and put $A_{i-1}A_{i+1}$ as a single link.

Reconfiguration problem and in particular, folding problem, been raised independently by several researchers. [3] has considered reconfiguration of robot arms inside a circle, with allowing its links to cross. In [4], Pei has proved that for a chain Γ inside a circle whose radius is sufficiently big, there is just one equivalence class when its links are allowed to cross. In [5] and [2], straightening a simple chain in the plane is studied and is proved that any simple chain can be straightened in the plane. And in [1] Arkin, Fekete and Mitchell have given an efficient algorithm to determined if a simple chain can be straightened by performing complete bend operations. In this paper, we study straightening a simple chain within a circle. we give a quadratic-time algorithm to straighten a simple unfoldable chain within a circle whose radius is sufficiently big. Then we prove that all of simple chains can

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be straightened within a circle, if and only if their straightened configuration can fit in the circle. Finally we show that any two configuration of these simple chains are equivalent.

2. Preliminaries

Let Γ be a simple chain inside circle $C(O,r)$ with joints $A_0...A_n$. For fitting straightened configuration of Γ in C, we must have $\sum_{i=1}^{n} l_i \leq 2r$. From now on, we suppose Γ is a simple chain inside C such that $\sum_{i=1}^{n} l_i < 2r$.

For a circle $C(O,r)$ and two points $x, y \in \partial C$, we use $\widehat{xy}$ to denote the clockwise arc from x to y. For a point $A_i \in \partial C$ we denote the other endpoint of the diameter of C which is containing A_i, by M_i.

Definition 1 *A joint A_{r_i} is called* rim joint *if it lies on boundary of circle C. We denote the set of all rim joints of chain Γ by $A_{Rim} = \{A_{r_0}, A_{r_1}, ..., A_{r_s}\}$.*

Definition 2 *For any rim joint A_i of chain Γ, the vector $\overrightarrow{OM_i}$ is called* radius vector *of A_i and is denoted by $\mathbf{r_i}$.*

Lemma 3 *There is a diameter $s = ab$ of circle C such that all of rim joints of chain Γ belong to one of the arcs $\widehat{ab}$ or $\widehat{ba}$.*

PROOF. If $A_{Rim} = \emptyset$, there is nothing to prove. Let A_i be a rim joint of Γ and X be a moving object which is walking along arc $\widehat{A_iM_i}$ clockwise, starting at the point A_i. Suppose A_r is the last rim joint of Γ that is visited by X. Diameter $s = A_rM_r$ is a solution. Because arc $\widehat{A_rM_r}$ contains no rim joint of Γ, except A_r. $\square$

Definition 4 *Suppose A_{Rim} has at least two point and rim joints of Γ belong to $\widehat{ab}$. The nearest rim joints to points a and b are denoted by A_f and A_e, respectively. These joints are called* limit-joints.

It is clear that all of the other rim joints of Γ are on arc $\widehat{A_fA_e}$.

Definition 5 *Let A_e and A_f be limit-joints of Γ. Vectors $\mathbf{r_e}$ and $\mathbf{r_f}$ are called* direction vectors.

Definition 6 *The sum of direction vectors, $\mathbf{r_e}$ and $\mathbf{r_f}$, is called* central direction *and denoted by $\mathbf{d}$, i.e., $\mathbf{d} = \mathbf{r_e} + \mathbf{r_f}$.*

Central Translation: We draw n vectors parallel to $\mathbf{d}$ from any joint A_i until hit circle at points N_i, then put $\varepsilon_i = ||\overrightarrow{A_iN_i}||$ and $\varepsilon = \min\{\varepsilon_i \mid 0 \leq i \leq n\}$. Translation of Γ inside C along the vector $\mathbf{d}_\varepsilon = \dfrac{\varepsilon}{||\mathbf{d}||}\mathbf{d}$, is called *central translation* of Γ. New positions of Γ and any joint A_i, after the central translation, are denoted by Γ' and A'_i. It is clear that $\varepsilon_e = \varepsilon_f$.

3. Unfoldable Simple Chains

Suppose $\sigma = (A_{i_1}, A_{i_2}, ..., A_{i_{n-1}})$ is an unfoldable sequence of joints of Γ. For straightening Γ inside circle $C(O,r)$, we propose the following algorithm which contains three steps:

Algorithm 1 *Unfolding Simple Chain Γ:*

step 1. $\Gamma' = \Gamma$; $j = 0$.
step 2. $\Gamma' = \Gamma'_U$; $j = j+1$. *If* $j = n$, *stop. else* $k = i_j$;
step 3. *Straighten A_k. Go to step 2.*

Now for step 3, straightening joint A_k within circle C, we propose the following algorithm which contains four steps:

Algorithm 2 *Straightening joint A_k:*

step 1. $\Gamma_0 = \Gamma[0,k]$; $\Gamma_n = \Gamma[k,n]$;
step 2. *Rotate Γ_0 about A_k until A_k straightens or one of joints of Γ_0 hits C and Γ_0 can not rotate more about A_k. If A_k straightens, stop; else go to step 3.*
step 3. *Rotate Γ_n about A_k until A_k straightens or one of joints of Γ_n hits C and Γ_n can not rotate more about A_k. If A_k straightens, stop; else go to step 4.*
step 4. *Calculate $\mathbf{d}_\varepsilon$ and transmit Γ by $\mathbf{d}_\varepsilon$. Go to step 2.*

4. Correctness of Algorithm 2

Any repeat of algorithm 2 is called a *phase* and the joint angle at A_k, at the end of phase i, is denoted by α_i. For showing correctness of algorithm 2, we show that during the algorithm, Γ remains simple and it remains inside C. And we prove that by using algorithm 2, after a finite number of repeats, A_k straightens. Furthermore, this finite number is independent of n.

Chain Γ remains simple, because A_k is an unfoldable joint in the plane. Now for showing that Γ remains inside C, we first prove that central translation always can be done.

Lemma 7 $\mathbf{d}_\varepsilon \neq \mathbf{0}$.

PROOF. If $\mathbf{d} = \mathbf{0}$, we have $\mathbf{r_f} = -\mathbf{r_e}$. That is implies $A_f = M_e$ and $A_e = M_f$, i.e., $A_fA_e = 2r$. Therefore $\sum_{i=1}^{n} l_i \geq \sum_{l_i \in \Gamma[e,f]} l_i \geq A_fA_e = 2r$. That is a contradiction. Thus $\mathbf{d} \neq \mathbf{0}$.

Because the angles between $\mathbf{d}$ and its components are less than $\pi/2$ and all of radius vectors lie between vectors $\mathbf{r_e}$ and $\mathbf{r_f}$, the angle between $\mathbf{d}$ and radius vectors are less than $\pi/2$, too. Thus any rim joint can transmit in direction $\mathbf{d}$ inside C. Any interior points of C also can transmit in all directions inside C. Therefore $\varepsilon \neq 0$. Consequently $\mathbf{d}_\varepsilon \neq \mathbf{0}$. □

It is clear that during the step 1 and step 2, all of the joints remain inside circle. At step 3, because $\varepsilon = \min\{\varepsilon_i \mid 0 \leq i \leq n\}$ and the angle between radius vectors and $\mathbf{d}$ are less than $\pi/2$, Γ remains inside C.
Now to show that after a finite number of repeats, algorithm 2 is terminated, we first show that at the end of any phase of algorithm 2, α_i becomes strongly close to π, i.e., $|\pi - \alpha_{i+1}| < |\pi - \alpha_i|$. So we have to prove at the end of central translation of Γ, at least one of the subchains Γ_0 or Γ_n, can rotate about A_k such that joint angle at A_k has became close to π. Note that at the end of step 1 and step 2, if A_k does not straighten, A_{Rim} has at least two points, one point from Γ_0 and the other point from Γ_n.

Lemma 8 *Let A_e and A_f be the limit-joints of Γ at phase i. If both of A_e and A_f belong to one of the subchains Γ_0 or Γ_n, then at the end of translation, none of the joints of the other subchain lie on ∂C. Furthermore, this subchain can rotate about A_k at phase $i+1$.*

PROOF. Assume without loss of generality that $A_f, A_e \in \Gamma_0$. Suppose for a contradiction, A_t is a joint of Γ_n such that $A'_t \in \partial C$. At the beginning of translation, Γ_n has at least one rim joint, A_m, which lies on arc $\widehat{A_fA_e}$. If A_k is in the exterior of closed curve $\delta = \widehat{A_fA_e} \cup \Gamma_0[e,f]$, $\Gamma_n[m,k]$ and $\Gamma_0[e,f]$ will be intersecting. Thus A_k is in the interior of δ. See figure 1. Therefore A'_k is in the interior of the closed curve $\delta' = \beta \cup \Gamma'_0[f,e]$ where

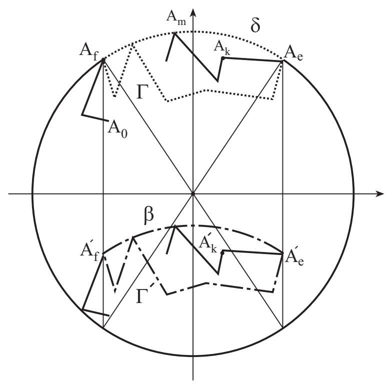

Fig. 1. *If Γ_n contains no limit-joints, Γ'_n has no rim joint.*

β is the translation of arc $\widehat{A_fA_e}$ by the vector $\mathbf{d}_\varepsilon$. But A'_t is in the exterior of δ'. So $\Gamma'_n[k,t]$ intersects boundary of δ'. That is a contradiction. Because $\Gamma_n[k,t]$ does not intersect boundary of δ. □

By lemma 8, we suppose A_e and A_f don't belong to the same subchain. From now on, the subchain which contains A_e is denoted by Γ_e and the other subchain which contains A_f is denoted by Γ_f. We have the following theorem.

Theorem 9 *At the end of translation, at least one of the subchains Γ_e and Γ_f can rotate about A_k.*

PROOF. Refer to full paper. □

Corollary 10 $|\alpha_i - \pi| > |\alpha_{i+1} - \pi|$.
By corollary 10, the configuration of Γ in two consecutive phase is different. Thus during the algorithm 2, straightening A_k is strongly progressed and cycling is not possible. Now for showing that after a finite number of repeats, algorithm 2 is terminated, we use simplicity of Γ. Assume without loss of generality that $\theta_k < \pi$. Thus according to definition of joint angle, Γ_0 must rotate about A_k clockwise. First suppose there is no confining region. So A_k can be straightened and then Γ_0 can rotate about A_k clockwise more, until first self-touching is occurred. This operation is called π-*passage motion* and the joint angle at A_k is denoted by $\pi + \tau_k$, that $\tau_k > 0$. Now suppose Γ is inside $C(O,r)$. We change the stopping criteria of algorithm 2, from achieving π to achieving $\pi + \tau_k$ and use this new algorithm on A_k. All of above proofs also hold for this new algorithm. So by corollary 10 we also have:

$$|\alpha_{i+1} - \pi - \tau_k| < |\alpha_i - \pi - \tau_k| \qquad (*)$$

Assumption $\theta_k < \pi$ yields: for every $i \geq 0$, $\alpha_i \leq \pi+\tau_k$. So $(*)$ yields $\pi+\tau_k-\alpha_{i+1} < \pi+\tau_k-\alpha_i$. In the other words, $\{\alpha_i\}_{i\geq 0}$ is a bounded and monotone sequence. Therefore it converges to its suprimum, $\pi + \tau_k$. Thus for every $\varepsilon > 0$ exists a finite natural number $N > 0$ such that for every $i \geq N$ we have $|\alpha_i - \pi - \tau_k| < \varepsilon$, i.e., for every $i \geq N$, $\pi+\tau_k-\alpha_i < \varepsilon$. Thus for $\varepsilon = \tau_k$, there is a finite number N_τ such that for all $i \geq N_\tau$, $\pi + \tau_k - \alpha_i < \tau_k$. So for $i = N_\tau$, we have $\pi + \tau_k - \alpha_{N_\tau} < \tau_k$, i.e., $\alpha_{N_\tau} > \pi$. Because N_τ is the smallest natural number that $\pi+\tau_k-\alpha_i < \tau_k$, we have $\pi+\tau_k-\alpha_{N_\tau-1} \geq \tau_k$, i.e., $\alpha_{N_\tau-1} \leq \pi$. Therefore A_k can straighten in phase N_τ or $N_\tau - 1$. Because $\alpha_{N_\tau-1} \leq \pi$ and $\alpha_{N_\tau} > \pi$. It is clear that N_τ is independent of n.

Therefore proof of correctness of algorithm 2 is terminated. Complexity of algorithm 1 is $O(n^2)$, because complexity of each step is $O(n)$ and the number of repeats is $n - 1$.

5. Arbitrary Simple Chains

Now we prove that an arbitrary simple chain can be straightened inside a circle. First we have the following theorem.

Theorem 11 *Any simple chain Γ can be straightened using a finite number of monotonic single-joint bend operations.*

PROOF. See [1] and [2]. □

Theorem 11 is true, if chain is inside a circle as a confining region.

Theorem 12 *A simple chain Γ can be straightened inside a circle using a finite number of monotonic single-joint bend operations, if* $\sum_{i=0}^{n} l_i < 2r$.

PROOF. By theorem 11, Γ can be straightened in the plane using a finite number of monotonic single-joint bend operations. If all of these bend operations are complete, Γ can be straightened by using algorithm 1. But if at least one of the bend operations is not complete, these bend operations will be in accordance with a sequence of motions, $M = \{M_j\}_{j=1}^{k}$, such that M_j is a partial bend operation and Γ can be straightened by using M. Any operation M_j is single-joint, so it is in accordance with a joint A_{i_j} and this accordance is not one to one, because M_j s are not complete. Suppose any bend operation M_j is changed joint angle at A_{i_j} by $\Delta(A_{i_j}; j)$. Now note that any operation M_j is monotone; so if we change the stopping criteria of algorithm 2, from achieving π to achieving $\theta_{i_j} + \Delta(A_{i_j}; j)$, this new algorithm can be used to perform any bend operation M_j inside C. Therefore Γ can be straightened inside C by performing M_j s in turn, because k is finite. □

6. Conclusion

Assume that Γ is a simple chain such that $\sum_{i=1}^{n} l_i < 2r$ and Γ_1 and Γ_2 are two configuration of Γ inside circle $C(O, r)$. We denote their straight configurations by L_1 and L_2, respectively. Let M be a sequence of bend operations for straightening Γ_2 inside C and M^R be the reverse of Motion M. It is clear that by performing M^R, L_2 can be reconfigured to Γ_2. Now by theorem 11, we can reconfigure Γ_1 to L_1, then we can reconfigure L_1 to L_2 by translation and rotation operations and finally we can reconfigure L_2 to Γ_2 by M^R. So Γ_1 can be reconfigured to Γ_2 inside C, i.e., for a simple chain Γ, if $\sum_{i=1}^{n} l_i < 2r$, then any two configurations of Γ, inside circle $C(O, r)$ are equivalent.

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