An Improved Bound for k-Sets in Three Dimensions

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1 Introduction

Let S be a set of n points in $\mathbb{R}^d$. A *k-set* of S is a subset $S' \subset S$ such that $S' = S \cap H$ for some halfspace H and $|S'| = k$. The problem of determining tight asymptotic bounds on the maximum number of k-sets is one of the most intriguing open problems in combinatorial geometry. Due to its importance in analyzing geometric algorithms [5, 9], the problem has caught the attention of computational geometers as well [3, 7, 8, 14, 16]. A close to optimal solution for the problem remains elusive even in the plane. The best asymptotic upper and lower bounds in the plane are $O(nk^{1/3})$ (see [6]) and $n \cdot 2^{\Omega(\sqrt{\log k})}$ (see [15]), respectively. In this paper we obtain the following result:

Theorem 1.1 *The number of k-sets in a set of n points in $\mathbb{R}^3$ is $O(nk^{3/2})$.*

This result improves the previous best known asymptotic upper bound of $O(nk^{5/3})$ (see Dey and Edelsbrunner [7] and Agarwal et al. [1]). The best known asymptotic lower bound for the number of k-sets in three dimensions is $nk \cdot 2^{\Omega(\sqrt{\log k})}$ (see [15]).

2 An Overview of our Technique

(a) The paper [1] presents a general technique, based on random sampling, for transforming an upper bound on the number of k-sets that is independent of k to a bound that does depend on k. Our main thrust will thus be to establish the upper bound $O(n^{5/2})$ for the number of k-sets. This, combined with the technique of [1], will imply Theorem 1.1.

(b) We assume that the set S is in general position, meaning that no four points in S lie in a common plane. Applying a small perturbation to the points of any set S yields a set of points in general position and the number of k-sets does not decrease.

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(c) We consider the set T of halving triangles spanned by S: A triangle $\Delta = abc$, with vertices $a, b, c \in S$ is a *halving triangle* if the plane containing Δ has the same number of points of S on either side. (Note that n has to be odd for halving triangles to exist, and we will indeed assume, without loss of generality, that n is odd.) We show that $|T| = O(n^{5/2})$. This implies that the number of k-triangles, for any k, is also bounded by $O(n^{5/2})$, where a *k-triangle* is a triangle Δ spanned by three points in S with exactly k points of S on one side of the plane containing Δ. Indeed, choose a direction d not contained in the plane of any k-triangle and add $|n-3-2k|$ extra points to S far enough in the direction d or $-d$. Each k-triangle in S turns into a halving triangle in one of the two resulting configurations. It is well known [2], that the $O(n^{5/2})$ bound on the number of k-triangles for any k carries over to the same bound on the number of k-sets.

(d) All the previous approaches are based on (the 3-dimensional extension of) Lovász Lemma [4]: Any line crosses (the relative interiors of) at most $O(n^2)$ halving triangles. The preceding techniques aimed to derive a general lower bound for the number of such crossings. Specifically, they showed that for any collection of t triangles spanned by the points of S there exists a line that crosses many triangles, where the best lower bound for this number of crossings is $\Omega(t^3/n^6)$ [7]. Combining this lower bound with the upper bound provided by Lovász Lemma, one obtains an upper bound of $O(n^{8/3})$ for the number of k-sets.

(e) In contrast, our technique focuses on the specific set T of halving triangles, and exploits the structure of this set. The main property of this set, which is also used in deriving Lovász Lemma, is the *antipodality* property, which we re-establish rigorously. Informally, it asserts that the halving triangles with a common edge pq alternate sides, as we rotate a plane about pq. See Figure 1 for an illustration of this property. This is the only property of the set T that is needed in the proof.

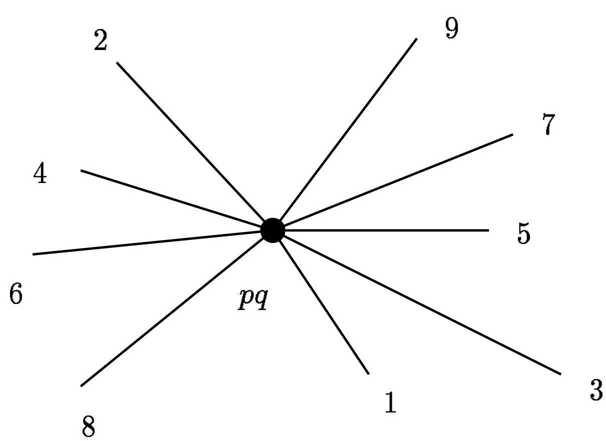

Figure 1: The antipodality property of halving triangles: The common edge pq is shown head-on, as a point; as we rotate a plane around pq we encounter the other endpoints of these triangles in the order shown.

(f) Our technique only considers interaction between pairs of triangles of T *with a common vertex*. Specifically, we consider *crossings* between such pairs of triangles, where two triangles pab and pcd cross each other if their relative interiors have a nonempty intersection (in this case p is the only common vertex of these triangles).

(g) Our proof proceeds by deriving both an upper bound and a lower bound on the number of triangle crossings (of the above special type) in T. The upper bound is $O(n^4)$ and it is an easy consequence of Lovász Lemma in 3-space. The lower bound is $\Omega(t^2/n)$, and is proven using arguments that extend those that were used in [6] for the analysis of k-sets in the plane. These upper and lower bounds immediately yield the desired bound on the number of k-sets.

(h) Concerning the upper bound $O(n^4)$: If pab and pcd is a pair of crossing triangles, then either ab crosses pcd or cd crosses pab. The number of crossings between a specific edge ab and the triangles of T is $O(n^2)$, by Lovász Lemma, and each such crossing materializes by at most 3 crossing pairs of triangles. This yields the $O(n^4)$ bound.

(i) The proof of the lower bound is more involved. For each point $p \in S$ we take a horizontal plane π_p slightly above p and project all triangles in T incident to p and having at least one other vertex above p centrally from p onto π_p. We obtain a straight-line graph G_p, consisting of segments and rays delimited by fewer than n points. We show that each G_p has a 'planar antipodality' property similar to the 3-dimensional property of T. This allows us to decompose G_p into *concave chains*, in a manner similar to that used by Agarwal et al. [1] and by Dey [6]. Using a counting argument, we then show that most pairs of chains are *crossing* (they have a pair of crossing edges). Any such crossing is induced by a crossing of two triangles of T incident to p. Hence, summing the number of pairs of chains in the graphs G_p, over all $p \in S$, yields the asserted lower bound of $\Omega(t^2/n)$ on the number of crossings in T.

3 Open problems

(a) Our analysis is based on the upper bound $O(n^4)$ on the number of crossings between triangles that share a vertex. However, this bound seems to be weak, because, for an edge ab connecting two points a, b of S, we want to count the number of k-triangles pcd that it crosses, with the additional constraint that pab is also a k-triangle. In our derivation we do not exploit this constraint at all, so the first open problem is whether this bound can be improved, taking into account this constraint.
(b) We conjecture that the following holds: Given a set S of n points in 3-space in general position and an arbitrary set T of t triangles spanned by S, there exists a line that crosses $\Omega(t^2/n^3)$ triangles of T. This bound is significantly larger than the bound $\Omega(t^3/n^6)$ of [7] and it would yield a trivial proof of Theorem 1.1 (using Lovász Lemma). We are not aware of any construction that contradicts this conjectured bound. This bound is best possible, for $t = \Omega(n^2)$, which can be shown by a simple construction.
(c) An even stronger conjecture is the following: Given a set S of n points in the plane in general position and an arbitrary set T of t triangles spanned by S, there exists a point that lies in $\Omega(t^2/n^3)$ triangles of T. The best known lower bound, due to [3], is $\Omega(t^3/(n^6 \log^5 n))$. Again, the conjectured bound is best possible for $t = \Omega(n^2)$. (Note that if (c) is true then the following strengthening of (b) also holds: Given S and T as in (b), then for any direction u there exists a line parallel to u that crosses $\Omega(t^2/n^3)$ triangles of T.)
(d) Finally, can the technique used in this paper be extended to higher dimensions?

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