Electromagnetic Scattering, 1D/2D Obstacle Problem, and Discrete Geodesics

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An important practical problem in communications is to numerically predict radiation directivity properties of antennas mounted on geometrically complex platforms acting as scatterers. High-frequency methods of electromagnetics have been used extensively for dealing with this problem in cases when the platform can be represented as a collection of a small number of simple shapes such as cylinders, ellipsoids, pieces of planes, cones, etc. When such a representation of the platform is sufficiently accurate the high-frequency methods provide an accurate estimate of the electric field produced by an antenna in a given direction. However, comparisons of computed results with experimental data showed that large errors may result in from inaccurate representation of the platform geometry. Consequently, there is a need for algorithms capable of performing high-frequency analyses in geometrically complex environments that can not be reduced to small collections of simple shapes.

The connection of the above problem to geometry in general, and computational geometry, in particular, is in that the high-frequency methods are based on geometrical optics and its extensions and, in order to apply these methods, one needs first to construct the propagation paths from the antenna (think of it as a point source!) to a given observation point at infinity specified as a point on a unit sphere[1]. Such observation point at infinity is usually referred to as the "far-field" direction. When the scatterer does not obstruct the wave propagation the path is a linear ray. Otherwise, the scatterer acts as an obstacle and the propagation path may have a nonempty contact set with the surface of the scatterer. In the present setting the scatterer is assumed to be perfectly conducting, and no penetration of the paths through the scatterer is allowed.

One of the main principles of the classical geometrical optics is that fields propagate along paths that are critical points of the Fermat functional. The Euler equations for these critical points lead to the eikonal equation. In classical setting the scatterer is smooth and usually only the paths that propagate in free space and reflect or refract are considered.

In contrast, in the geometric theory of diffraction and its extensions (especially, the uniform theory of diffraction) the notion of a propagation path is

[1] Typically, an engineer may specify a set of sample directions ("pattern cut") in which the electric field produced by an antenna is required.

generalized to include paths not only with spatial segments and reflections, but also those that diffract at geometric singularities such as corners and edges, may attach to the surface of the scatterer, "creep" along it, then detach, diffract, and so on.

If a propagation path from the source to an observation point is known, the field at the observation point due to that path can be obtained by "gluing" together the local scattering information along the path. In principle, this scattering information can be computed by high-frequency techniques, though application of these techniques is by no means straight-forward. Finally, the total field at the observation point is obtained as a vector sum of contributions by the fields propagating along individual paths from the source to that observation point.

Let us now state the geometric problem of finding the required propagation paths in a more precise manner.

First of all, the surface of the scatterer, denoted by S, is assumed to be a closed, embedded, and oriented polyhedral surface in R^3, triangulated so that any edge belongs to exactly two triangles and the intersection of any two triangles is either an edge, a vertex, or empty. The orientation is always assumed to be outward. Note that S is not assumed to be convex and, in fact, may have genus > 0. However, for propagating paths which have a nonempty contact set with the surface of the scatterer the available high-frequency methods allow to calculate electric fields only when the scatterer is locally convex (but not necessarily strictly convex) at points of the contact set.

Denote by Ω the open subset of R^3 bounded by S.

Let l be a curve given by a piece-wise differentiable vector function

$$\gamma(\sigma) = (x(\sigma), y(\sigma), z(\sigma)), \quad \sigma \in [0, \infty), \tag{1}$$

where σ is the arc length. The Fermat functional is given by

$$\mathcal{F}(\gamma_0, \gamma_1) = \int n d\sigma, \tag{2}$$

where $n = const$ is the refractive index, $d\sigma$ is the arc length, and the integral is taken over the piece of l between points γ_0 and γ_1.

A curve (1) is "admissible" for the Fermat functional if the following conditions are satisfied:

$$\gamma \quad \text{is simple}, \tag{3}$$

$$\gamma(0) = R, \tag{4}$$

where R is a point where the radiation source is located[2],

$$\frac{d\gamma}{d\sigma} \longrightarrow \hat{t} \quad \text{as} \quad \sigma \longrightarrow \infty, \tag{5}$$

[2]It is assumed here that an antenna is represented as a point source on or off the surface of the scatterer.

where $\hat{t}$ is a unit vector denoting a far-field direction, and

$$\gamma(\sigma) \cap \Omega = \emptyset \quad \text{for any} \quad \sigma \in [0, \infty). \tag{6}$$

Denote by $\mathcal{R}(S, R, \hat{t})$ the collection of admissible curves for the scatterer S, source R, and far-field direction $\hat{t}$. A path $\gamma \in \mathcal{R}$ is called stationary for the Fermat functional $\mathcal{F}$ if

$$\delta \mathcal{F} = 0, \tag{7}$$

where δ is the variation of $\mathcal{R}$.

Because the curve l has infinite length the last condition should be understood in the following sense: for any two points A and B on a stationary curve γ the variation of the Fermat functional among curves from $\mathcal{R}$ which also pass through A and B and are sufficiently close to γ must be equal to zero.

Finally, a curve $\gamma \in \mathcal{R}$ is said to be locally minimizing (or, equivalently, a propagation path) if each point on it is an interior point of some sub-segments AB of the path for which the value of the Fermat functional does not exceed the value of that functional on any curve from $\mathcal{R}$ which also passes through A and B, coincides with γ outside the segment AB, and is sufficiently close to γ.

Note that a locally minimizing curve in $\mathcal{R}$ is also stationary.

Now, the formal statement of the problem is as follows. *For a given scatterer S, a point source R on or off the scatterer S, and a given far-field direction $\hat{t}$ find the locally minimizing curves of the Fermat functional in the class $\mathcal{R}(S, R, \hat{t})$.* For brevity, we will refer to this problem as the *discrete radiation* problem.

In certain cases, in order to capture paths with specific properties the variations may have to be further restricted. For example, in order to obtain paths that contain a reflection point on the scatterer one should require that admissible paths have a non-empty contact set on the scatterer. Similarly, in order to obtains paths diffracting at corners or, in some special cases at edges, one needs to further restrict classes of admissible variations. The definitions of stationary and locally minimizing curves in such cases must be modified accordingly.

Simple variational arguments show that for a propagation path the contact set may consist of isolated points and segments of surface geodesics. It may also be empty.

In applications related to directivity properties of antennas the diffracted field along creeping segments of a propagation path within the shadow region decays exponentially. Therefore, it is important to construct only the "dominant" paths and avoid searching for paths with contributions to the electric field below some user-specified level.

A heuristic algorithm for building the dominant paths in the discrete radiation problem has been developed and implemented. The algorithm starts with building a possibly large set of initial paths that are quasi-geodesics in the metric of the surface S. The initial paths are terminated at the shadow boundary (relative to the current far-field direction). For a given far-field direction the algorithm uses the local optimality condition at the shadow boundary to filter out some of the initial paths. In "generic" cases this procedure filters out most of the initial paths. The initial paths that remain are optimized. The optimization of each of the paths uses a a variant of unfolding strategy combined with lifting of paths off the surface S whenever possible. Experimental data obtained by running the algorithm on geometrically complex surfaces will also be presented and discussed.

Some related results on electromagnetics can be found in [1] and on discrete geodesics in [2], [3].

References

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