On the equipartitions of convex bodies and convex polygons *

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Abstract

In this paper we consider the equipartitions of plane convex bodies by means of chords, i. e. partitions produced by chords in such a way that all resulting pieces have equal area. We prove two basic results about the combinatorial structure of the pieces and the chords.

The problem of partitioning plane convex bodies into equal area pieces has been object of extensive research, especially from the pure geometry viewpoint [4, 6, 7, 8, 10, 13, 16], but also regarding the algorithmic issues [5, 15]. Extensions to non-convex bodies, simultaneous partition of area and perimeter, and discrete weights for the pieces, have also been considered [1, 2, 3, 14].

In this work we study the partitions determined by means of chords. The first question that arises is to know when there exists a solution, that is to say which properties have the sets of chords that divide the convex set into subsets of the same area. When the body is not a triangle and the chords are non crossing, we prove the following result:

Theorem 1 *The dual tree of a partition of a plane convex body, different from a triangle, by means of m non crossing chords, into $m+1$ equal area pieces, is a path.*

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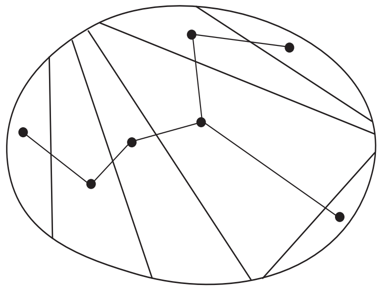

Fig. 1 Not all the pieces in the partition have equal area, because the dual tree is not a path.

We equivalently prove that any partition of a convex body, different from a triangle, by a set of non crossing chords, has at most two extreme regions, i.e. regions bounded by the boundary of the body and only one chord.

When the object is a triangle the theorem also applies, with the only exception of the partition into four pieces induced by the three segments connecting midpoints of the sides. This fact gives the basic argument in the proof of the general theorem.

The existence of partitions of convex sets into equal area pieces, by a set of chords with a common point interior to the object, has been studied in connection with measures of symmetry of plane convex bodies [10]. It has been shown that for $m \leq 3$ chords there exists always a solution. This result is obtained as a consequence of the Borsuk-Ulam theorem for $m \leq 2$. E. F. Buck and R. C. Buck [4] proved that for each plane convex body K there exists at least three straight lines with a common point such that each of the six wedges determines a piece of K with area equal to one-sixth of the area of K. Such a point is called a *six-partite point* of K. Eggleston studied some properties of these points and proved that a given convex set may have more than one six-partite point [6, 7], in fact it might even have a set of six-partite points with positive linear measure.

No characterization of convex sets that admit more than three lines through a common interior point, dividing the body into subsets of equal area, seems to be known.

We show here that there is no solution for the set of non centrally symmetric convex n-gons for $m > n$ chords. When the problem has solution the straight lines determined by the chords are area-bisectors of the polygon, that is, each of them cuts it into two pieces of equal area. Let $n(p)$ be the number of area-bisectors of P through a point $p \in P$. The existence of polygons such that there is an interior point p with a given number of bisectors through p, has been considered by Goldberg [8]. We have shown that $\max\{n(p) \mid p \in P\}$ is finite and at most equal to the number of inflection points of the dual curve of the family of lines that bisect the polygon [11, 12, 5, 2], which is less or equal than the number of vertices of the polygon.

Proposition 2 *Given a convex n-gon P, not centrally symmetric, let S be a set of chords with a common interior point to P that divide P into subsets of equal area, and let c be the number of inflection points of the dual curve of the family of lines that bisect P. Then $cardS \leq c \leq n$.*

As an immediate consequence we get:

Theorem 3 *A convex n-gon P admits no equipartition by means of $m > n$ chords through a common point interior to P.*

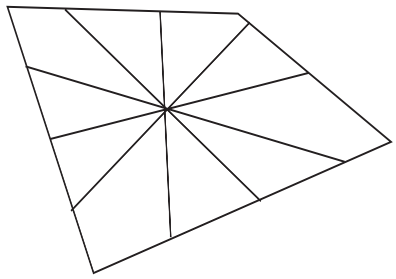

Fig. 2 Not all the pieces in the partition have equal area, because the number of chords is greater than the number of polygon edges.

More geometric results, as well as algorithms for existence and total length minimization, for some specific configurations, are described in [9]. It is worth mentioning that numerical troubles arise, even for a combinatorially fixed situation, when there is no unicity of the solution.

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