DYNAMIC LAGUERRE DIAGRAMS MADE EASY

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Abstract. We present a dynamic algorithm for construction and maintenance of Laguerre (or Power Voronoi) diagrams. We prove an "empty circle criterion" that characterises the dual of the Laguerre diagram, like the emty circumcircle criterion characterizes the Dealaunay triangulation. We use a data structure similar to the Delaunay tree in order to maintain the dynamic Laguerre diagram in $O(\log n)$ expected running time per operation.

1. Introduction

The Voronoi diagram has been generalized to different sets of sites (lines, circles, curves, polygons), and different metrics or geometries (L_p metrics, convex distances, Laguerre geometry). The generalization of the Voronoi diagram to the Laguerre geometry is known as Laguerre diagram [9]. Laguerre diagrams are generalized Voronoi diagrams based on a metric different from the Euclidean distance: the power of a point relatively to a weight circle. This Laguerre metric is intrinsically connected to the inversive geometry and extends to the power of a point relatively to a sphere in three dimensions (or relatively to hypersheres in higher dimensions). The difference between the Laguerre diagram and the Power Voronoi diagram is that in the former, the radii of the weight circles correspond to the absolute value of the elevation of the sites in the three-dimensional space (see Imai and *al.* [8]), meanwhile in the later, they correspond to the square root of the weight of the site (see Aurenhammer [1]).

The work of Edelsbrunner and Shah [6] provides an algorithm for the incremental construction of Power Voronoi diagrams in the d-dimensional space by lifting the sites to the $d+1$-dimensional paraboloid, constructing the convex hull in the $d+1$-dimensional space, and projecting it back to the d-dimensional space. The projection of this convex hull onto the d-dimensional space is the Power Voronoi diagram of the initial set of sites. Their algorithm runs in $O\left(n\,log\,n + n^{\lceil \frac{d}{2} \rceil}\right)$ expected time. While the lifting of the sites to the $d+1$-dimensional paraboloid and the projection to the d-dimensional space does not influence the efficiency of a static algorithm (they have linear time efficiency), it is not conceivable in the case of an online algorithm (the efficiency would be quadratic even in the average case). The algorithm proposed by Gavrilova [7] converts incrementally ordinary Voronoi diagrams into Power Voronoi diagrams in $O\left(n^2\right)$ worst-case running time without lifting to the d+1-dimensional space. Her work [7] does not provide a geometric construction for this conversion. Moreover her assumption (Assumption 3.1.1 in [7]), that the weight circles do not intersect is very limitative, since in the applications the circles are not guaranteed to be disjoint.

In this paper, we will study an $O\left(n^2\right)$ worst case running time and $O(\log n)$ expected running time per operation algorithm for the construction and maintenance of Laguerre diagrams. This algorithm has a linear storage cost. We exhibit a property of the Voronoi diagram in the Laguerre geometry analogous to the empty circumcircle property of the dual of the ordinary Voronoi diagram (the Delaunay triangulation [10]). This work extends the applicability of an "empty circle criterion" without assuming that the circles do not intersect. This is particularly useful when there is a need for the dynamic maintenance of the topology induced by the Laguerre diagram. By this topology we mean the spatial adjacency relationships among sites defined as Voronoi regions of two sites sharing a common Voronoi edge. Moreover, we give a geometric and an algebraic interpretations of the "empty circle criterion" for the geometric dual of the Laguerre diagram. The algorithm for the construction and maintenance of Laguerre diagrams uses a hierarchical data structure analogous to the Delaunay tree [10], that stores all the successive versions of the dual of the Laguerre diagram during the insertion process.

2. Preliminaries

Let $\mathbb{R}^2$ be the Euclidean plane. In this paper, we denote by $\langle a,b \rangle$ the line passing through the points a and b [2]. Let m be a point of $\mathbb{R}^2$. Let $S(a,r)$ be a circle of centre a and radius r. Its equation is given by $S(x) = 0$ where $S(x) = d(a,x)^2 - r^2$ (see [4]). The power of the point m with respect to the circle S is defined as [4]: $S(m) = d(a,m)^2 - r^2$. If $S(m) = 0$, the point m is on $S(a,r)$, if $S(m) < 0$, the point m is inside $S(a,r)$, if $S(m) > 0$, m is outside $S(a,r)$. If we take any line l passing through m and intersecting $S(a,r)$, at the intersection points t and t', we have the following property [4]: $\overrightarrow{mt} \cdot \overrightarrow{mt'} = S(m)$. Now, let's consider two circles $S(a,r)$ and $S'(a',r')$. The locus of points whose power with respect to $S(a,r)$ equals their power with respect to $S'(a',r')$ is called the radical axis or

Key words and phrases. Laguerre diagrams, dual of the Laguerre diagram, empty circle criterion, dynamic algorithm.

chordale of $S(a,r)$ and $S'(a',r')$. This locus is defined if, and only if $a \neq a'$ (see Berger [2]). In the case where it is defined, its equation is: $S(x) = S'(x)$, and it is orthogonal to $\overrightarrow{aa'}$ [2].

Two circles $S(a,r)$ and $S'(a',r')$ are said to be orthogonal if, and only if, the following equivalent conditions hold:

1. $S'(a) = r^2$
2. $S(a') = r'^2$
3. $d(a,a')^2 = r^2 + r'^2$ [2].

Let $\mathcal{S} = \{S_i\}$ be the set of generators (circles), where S_i is the circle whose centre is the projection $\overrightarrow{p_i}$ of the site onto the two-dimensional plane Oxy, and its radius (the absolute value of the elevation of the site) z_i [8]. The Laguerre region of S_i is the set of points of the Euclidean plane whose power with respect to S_i is smaller than or equal to their power with respect to the other spheres [8]. It is defined by:

$\mathcal{V}(S_i) = \left\{M \in \mathbb{R}^2 : \forall j \neq i, S_i(M) \leq S_j(M)\right\}$.

The bisector H_{ij} between S_i and S_j is the locus of points whose power with respect to S_i equals their power with respect to S_j. Therefore, H_{ij} is the radical axis or chordale of S_i and S_j.

3. The "empty circle criterion"

In this section, we will study the "empty circumcircle criterion" for the geometric dual of the Laguerre diagram, that works even if the weight circles intersect. **In the rest of this paper, we suppose that any three sites a, b, and c are not aligned.** Let us consider a triple of circles $A(a,\alpha)$, $B(b,\beta)$, and $C(c,\gamma)$ (see the bold circles on Figures 1 and 2) of the set of circles $\mathcal{S}$. Let $D(d,\delta)$ be a circle intersecting the circles $A(a,\alpha)$, $B(b,\beta)$, and $C(c,\gamma)$ (see the dashed circle on Figure 1). The ordinary Voronoi vertex of the triple of points $\{a,b,c\}$ is the point of the plane equidistant from these three points, and therefore, it is the centre of the circle passing through these three points. The circle passing through the three points a, b, and c is called the Delaunay circle in this section. In the algorithm, we use the Delaunay circle as the circle $D(d,\delta)$ (see the dashed circle on Figure 2).

Lemma 3.1. *Let H_{AD} denote the chordale of A and D, H_{BD} denote the chordale of B and D, and H_{CD} denote the chordale of C and D. Then, they intersect two by two at three points i, j, and k (see Figures 1 and 2).*

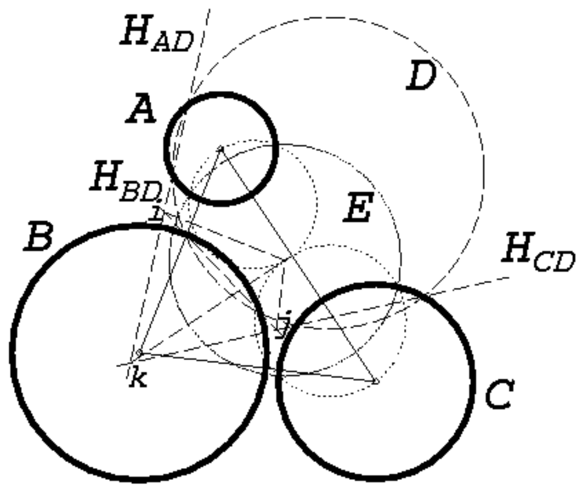

FIGURE 3.1. The "Laguerre empty circle, the "weight" circles and the auxiliary circle D

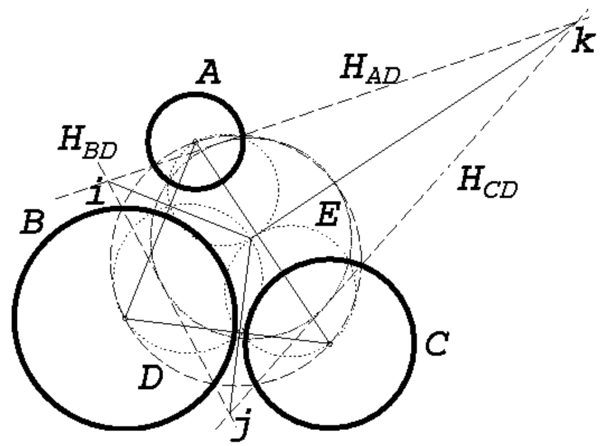

FIGURE 3.2. The "Laguerre empty circle", the "weight" circles and the Delaunay empty circle

Proof. By definition, the radical axis of two circles is orthogonal to the line joining the centres of the two circles. Therefore, $H_{AD} \perp \langle a,d\rangle$, and $H_{BD} \perp \langle b,d\rangle$, and $H_{CD} \perp \langle c,d\rangle$. Since we supposed that a, b, and c are not aligned, $\langle a,d\rangle$, $\langle b,d\rangle$, and $\langle c,d\rangle$, which all contain d, are three distinct lines with distinct directions. Therefore, since $H_{AD} \perp \langle a,d\rangle$, and $H_{BD} \perp \langle b,d\rangle$, and $H_{CD} \perp \langle c,d\rangle$; H_{AD}, H_{BD}, and H_{CD} are three distinct lines with distinct directions. Therefore, they intersect two by two at three points i, j, and k. $\square$

Lemma 3.2. *Let i denote the intersection of H_{AD} with H_{BD}, j denote the intersection of H_{BD} with H_{CD}, and k denote the intersection of H_{CD} with H_{AD}. Then, the line orthogonal to $\langle a,b\rangle$ and containing i, the line orthogonal to $\langle b,c\rangle$ and containing j, and the line orthogonal to $\langle c,a\rangle$ and containing k, intersect at a common point p (see Figures 1 and 2).*

Proof. Since i pertains to H_{AD}, $A(i) = D(i)$. Since i pertains also to H_{BD}, $D(i) = B(i)$. Therefore, $A(i) = D(i) = B(i)$. Therefore i pertains to the radical axis H_{AB} of A, and B. Similarly, j pertains to the radical axis H_{BC} of B, and C; and k pertains to the radical axis H_{AC} of A, and C. By definition, the radical axis of two circles is orthogonal to the line joining the centres. Therefore, the line orthogonal to $\langle a, b\rangle$ and containing i is the radical axis H_{AB} of A, and B, and its equation is $A(x) = B(x)$. Using the same argument based on the definition of the radical axis of two circles, we can state that the line orthogonal to $\langle b, c\rangle$ and containing j is the radical axis H_{BC} of B, and C, and its equation is $B(x) = C(x)$; and the line orthogonal to $\langle a, c\rangle$ and containing k is the radical axis H_{AC} of A, and C, and its equation is $A(x) = C(x)$. Since a, b, and c are not aligned, $\langle a, b\rangle$ and $\langle b, c\rangle$ are not parallel. Therefore, since $\langle a, b\rangle \perp H_{AB}$ and $\langle b, c\rangle \perp H_{BC}$, H_{AB} and H_{BC} are not parallel. Thus, they have an intersection point. Let's call it p. Since p pertains to H_{AB} (whose equation is $A(x) = B(x)$), $A(p) = B(p)$ holds. Since p pertains to H_{BC} (whose equation is $B(x) = C(x)$), $B(p) = C(p)$ holds. Therefore, $A(p) = B(p) = C(p)$, which together with the conclusion of this proof implies the next corollary. Since $A(p) = C(p)$, p belongs to H_{AC} (whose equation has been shown to be $A(x) = C(x)$). Therefore, p belongs to H_{AB}, H_{BC}, and H_{AC}. Therefore, p belongs to the line orthogonal to $\langle a, b\rangle$ and containing i, the line orthogonal to $\langle b, c\rangle$ and containing j, and the line orthogonal to $\langle c, a\rangle$ and containing k. □

Corollary 3.3. *The Laguerre vertex of $A(a, \alpha)$, $B(b, \beta)$, and $C(c, \gamma)$ is the point p intersection of the three lines mentioned in Lemma 2.*

Theorem 3.4. *The Laguerre vertex of $A(a, \alpha)$, $B(b, \beta)$, and $C(c, \gamma)$ is the intersection of the internal bisectors of the three radical axes H_{AD}, H_{BD}, and H_{CD}.*

Proof. The radical axis H_{AD} is orthogonal to $\langle a, d\rangle$. The radical axis H_{BD} is orthogonal to $\langle b, d\rangle$. Therefore the angle $\overline{H_{AD}, H_{BD}}$ formed by the chordales H_{AD} and H_{BD} is the same as the angle $\overrightarrow{da}, \overrightarrow{db}$ formed by the lines $\langle d, a\rangle$ and $\langle d, b\rangle$. Since d is the circumcentre of a, b, and c, the triangle $\{a, d, b\}$ is isocele in d ($da = db$). Therefore, the line l' orthogonal to $\langle a, b\rangle$ and containing d is also the bisector of the angle $\overrightarrow{da}, \overrightarrow{db}$. We have proved in proposition 1, that the Laguerre vertex pertains to the line l orthogonal to $\langle a, b\rangle$ and containing i. This line l is thus parallel to l' (they are both orthogonal to $\langle a, b\rangle$), and thus l bisects internally the $\overline{H_{AD}, H_{BD}}$ in i. Therefore, the Laguerre vertex pertains to the internal bisector of the angle $\overline{H_{AD}, H_{BD}}$. From Proposition 1, we know that the Laguerre vertex pertains also to the line orthogonal to $\langle b, c\rangle$ and containing j, and to the line orthogonal to $\langle c, a\rangle$ and containing k. Therefore, using a similar reasoning, we can prove that the Laguerre vertex pertains also to the internal bisector of the angle $\overline{H_{AD}, H_{BD}}$, and the internal bisector of the angle $\overline{H_{AD}, H_{BD}}$. Therefore, the Laguerre vertex is the intersection of the internal bisectors of the three radical axes H_{AD}, H_{BD}, and H_{CD}. □

Theorem 3.5. *There exists a (possibly pure imaginary) circle centered at the Laguerre vertex of $A(a, \alpha)$, $B(b, \beta)$, and $C(c, \gamma)$ and orthogonal to both $A(a, \alpha)$, $B(b, \beta)$, and $C(c, \gamma)$. We call it the Laguerre empty circle of $A(a, \alpha)$, $B(b, \beta)$, and $C(c, \gamma)$ (shown as circle E in plain on Figures 1 and 2).*

Proof. We know that $A(p) = B(p) = C(p)$. This implies that $d(p, a)^2 - \alpha^2 = d(p, b)^2 - \beta^2 = d(p, c)^2 - \gamma^2$. Let ρ be $d(p, a)^2 - \alpha^2 = d(p, b)^2 - \beta^2 = d(p, c)^2 - \gamma^2$. Then, if we denote by ι the real or pure imaginary number such as $\iota^2 = \rho$, the following relationships hold: $d(p, a)^2 = \alpha^2 + \iota^2$, $d(p, b)^2 = \beta^2 + \iota^2$, and $d(p, c)^2 = \gamma^2 + \iota^2$. Therefore the circle centered at p and of radius ι is orthogonal to both $A(a, \alpha)$, $B(b, \beta)$, and $C(c, \gamma)$. If this circle is a pure imaginary circle, the orthogonality does not have a geometric interpretation [3]. However, in all the cases (real or pure imaginary circle), the orthogonality is well defined in the sense of the orthogonality in the space of generalized spheres (i.e. the orthogonality defined by the bilinear symetric form associated with the fundamental quadratic form of the space of generalized spheres [3]). If the circle is a real circle, the algebraic definition of orthogonality coincides with its geometric definition. This circle is a pure imaginary circle [3] if, and only if, $A(p) = B(p) = C(p) < 0$. □

Corollary 3.6. *This circle is a true circle if, and only if, the intersection of the disk bounded by A, and the disk bounded by B, and the disk bounded by C is empty.*

4. The dynamic algorithm for constructing Laguerre diagrams

Lemma 4.1. *Let e be the vertex of the triple $A(a, \alpha)$, $B(b, \beta)$, and $C(c, \gamma)$, and $E(e, \epsilon)$ be the circle orthogonal to $A(a, \alpha)$, $B(b, \beta)$, and $C(c, \gamma)$. Then, there exists a circle $D(d, \delta)$ of $\mathcal{S}$ for which the vertex e is also the vertex of the triples $\{A, B, D\}$ and $\{A, C, D\}$ if, and only if, $D(d, \delta)$ is orthogonal to $E(e, \epsilon)$.*

Proof. Only if: If such circle $D(d, \delta)$ exists, then $A(e) = B(e) = D(e)$ and $A(e) = C(e) = D(e)$. Therefore, $A(e) = B(e) = C(e) = D(e)$. Since $E(e, \epsilon)$ is orthogonal to $A(a, \alpha)$, $B(b, \beta)$, and $C(c, \gamma)$, according to section 2, $A(e) = \epsilon^2$. Since $A(e) = D(e)$, we have $D(e) = \epsilon^2$, which expresses that $D(d, \delta)$ is orthogonal to $E(e, \epsilon)$.

If: If $D(d, \delta)$ is orthogonal to $E(e, \epsilon)$, then $D(e) = \epsilon^2$.

Since $E(e,\epsilon)$ is orthogonal to $A(a,\alpha)$, $B(b,\beta)$, and $C(c,\gamma)$, the following relationships hold: $A(e)=\epsilon^2$, $B(e)=\epsilon^2$, and $C(e)=\epsilon^2$. Since we had already $D(e)=\epsilon^2$, we get $A(e)=B(e)=C(e)=D(e)$. We can conclude that the vertex e is also the vertex of the triples $\{A,B,D\}$ and $\{A,C,D\}$ if, and only if, $D(d,\delta)$ is orthogonal to $E(e,\epsilon)$. □

Theorem 4.2. *Let e be the vertex of the triple $A(a,\alpha)$, $B(b,\beta)$, and $C(c,\gamma)$. Then, the vertex e is valid (the triangle $\{A,B,C\}$ pertains to the dual of the Laguerre diagram) if, and only if, the power of e with respect to any circle $D(d,\delta)$ of $\mathcal{S}$ is greater than any of the power of e with respect to $A(a,\alpha)$, $B(b,\beta)$, or $C(c,\gamma)$.*

Proof. The vertex e is valid (the triangle $\{A,B,C\}$ pertains to the dual of the Laguerre diagram) if, and only if, $A(a,\alpha)$, $B(b,\beta)$, and $C(c,\gamma)$ are the three sites closest to e in the Laguerre metric. Since, e has the same power with respect to $A(a,\alpha)$, $B(b,\beta)$, and $C(c,\gamma)$; $A(a,\alpha)$, $B(b,\beta)$, and $C(c,\gamma)$ are the three sites closest to e in the Laguerre metric if, and only if, there is no other site $D(d,\delta)$ of $\mathcal{S}$, that is closer to, or at the same distance from e than $A(a,\alpha)$, $B(b,\beta)$, or $C(c,\gamma)$ in the Laguerre metric. Therefore e is valid if, and only if, the power of e with respect to any circle $D(d,\delta)$ of $\mathcal{S}$ is greater than any of the power of e with respect to $A(a,\alpha)$, $B(b,\beta)$, or $C(c,\gamma)$. □

The proofs that have been given in this section are much simpler than the proof of the INCIRCLE test in Gavrilova [7]. Moreover, the INCIRCLE test in [7] assumes that no two weight circles intersect. Lemma 3 gives a necessary and sufficient condition and a geometric interpretation for the degenerate case where a Laguerre vertex is closer to four circles. Theorem 4 gives a way to test if a Laguerre vertex is valid. Its geometric interpretation is that the vertex e is valid (the triangle $\{A,B,C\}$ pertains to the dual of the Laguerre diagram) if, and only if, no circle $D(d,\delta)$ of $\mathcal{S}$, is orthogonal to a circle centered at e and of radius smaller than or equal to ϵ. The assumption that no three circle centres are aligned guarantees that the bisectors (the chordales) are defined and intersect, and that each Laguerre vertex is three-valent.

When a new point is added, we locate the triangle T in which it lies, then we connect this new point to the triangulation by replacing T by three new triangles whose vertices are the vertices of T and the new point. Then we check every circle tangent to the weight circles of the points of every new triangle. If a triangle switch has to be performed, we perform the same check for all the tangent circles corresponding to the triangles generated by the triangle switch. In the worst case, $O(n)$ tests have to be performed for each insertion. We use a data structure similar to the Delaunay tree [5], that stores the sequences of the dual of the Laguerre diagram corresponding to our above mentioned insertion process, and that requires a linear amount of storage [5]. The deletion of a site is performed by removing all the edges attached to the site being removed, reconstructing the data structure as if the site was never inserted, and then retriangulating the hole created by the removal of the edges. The randomized analysis gives an expected logarithmic running time per operation.

5. Conclusions

We have presented a dynamic algorithm for constructing Laguerre diagrams. We have presented and proved the empty circle criterion for the dual of the Laguerre diagram. Our empty circle criterion is valid even if the weight circles intersect. Our algorithm generalizes the results in [7]. Even though this algorithm complexity is not better than the one of lifting based algorithms, the computations involved are simpler (only determinants), the topology is known directly and dynamically maintained.

6. Acknowledgments

This research work has received the financial support of the Natural Sciences and Engineering Research Council of Canada (NSERC, to both authors).

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