The face-wise continuity in hot wire cutting of polyhedral sets

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Abstract

We study the problem of cutting out polyhedral sets with a hot wire cutter. Hot wire cutters are popular tools for cutting expanded polystyrene. They do this by heating a thin wire with a power supply. The wire is mounted on a mobile frame that moves above the stationary block of foam during the cutting process. We assume the most general case where the wire has three degrees of freedom. Our interest is to study the feasibility of cutting out an object under a variety of constraints as well as to provide a framework for CAD software controlling such cutters. We review a number of problems that arise in hot wire cutting of (non-convex) polyhedral sets. Although manufacturing is the primary subject, the paper very strongly relates to other aspects of computational geometry. Most prominently, it appears that the hot wire cutting is a very natural application for problems concerning sets of lines in two and three dimensions. In this paper we focus on algorithmic issues of cutting out (non-convex) polyhedral sets maintaining face continuity. That is, we are asking when each face of the polyhedral set can be cut out in one sweep of the wire. In an abstract setting the face-wise continuity is related to f-stabbing for a set of polygons $f, f_1, \ldots, f_k$ where we want to stab each point of f while omitting $f_1, \ldots, f_k$.

We provide an $O(n \log n)$ algorithm for constructing for each face a hot wire cutting path with face-wise continuity, if it exists and where n is the total number of edges in the polyhedral set. The algorithm is also able to identify discontinuities (places where the tool must be detached and repositioned) if a continuous path does not exist. In $O(n^2 \log n)$ time the algorithm constructs a graph of face continuous cuts for the entire polyhedral set; based on the graph we can plan cutting paths under a variety of constraints. We also show that even if individual faces can be cut out with the face-wise continuity, the problem of planning a cutting path for the entire polyhedral set which visits each face only once is NP-complete. This holds even for polytops and follows from properties of cubic graphs.

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1 Background and definitions

Hot wire cutters are tools for cutting styrofoam (expanded polystyrene). They do this by heating a thin wire with a low voltage power supply. The wire is mounted on a mobile frame (gantry) that moves above the stationary block of foam during the cutting process. The block sits on a table. More advanced models are equipped with a turntable so the block of styrofoam can be rotated which allows for cutting shapes with rotational symmetry. Two ends of the wire are attached to the opposite sides of the frame and they can move vertically on the frame. When the gantry moves, the hot wire slices through styrofoam like a hot knife through ice cream. Foam hot wire cutters are used for a variety of applications ranging from serious manufacturing jobs to crafts and hobbies. Products that can be manufactured with hot wire cutters include reverse molds for the concrete industry, packaging projects, signs and retail graphics, props and exhibit stands, pre-production prototypes, packaging molds, architectural and building components (EIFS), concrete casting molds, foam props for retail designs, property gateways and entrances, movie sets and stage props, pipe insulation, hobbies (such as railroad models), and craft projects.

This paper presents one aspect of the hot wire cutting and is a part of the broad study of algorithmic aspects of wire-cutting. The general goal of this study is laying a ground for CAD software in the next advanced generation of cutters where the ends of the wire can move independently on the arms of the frame and the frame can rotate, which provides three degrees of freedom for the wire. Our results apply to this general setting as well as to simpler cutters with a limited degree of freedom.

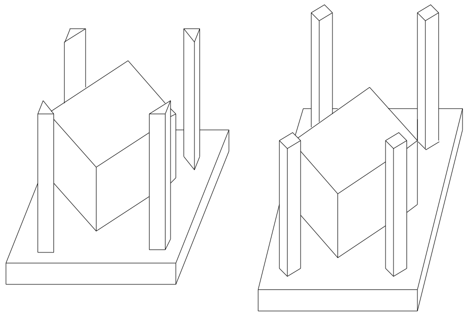
To better understand the problem, let us look at examples that illustrate various aspects and difficulties related to hot wire cutting. Let us focus on feasibility of cutting and on aspects, imposed by technological limitations, related to single versus multiple cutting of the faces.

Questions that can be studies in this context are:

1. feasibility - what shapes can/cannot be cut out?
2. face-wise continuity - can each face be cut out without moving the wire away from the plane containing this face and then returning to this plane?
3. polyhedron-wise continuity - can the polyhedral set be cut out without disconnecting and reattaching the wire (e.g., a cube with a hole through the middle)?
4. admissible moves for the wire - is it necessary that the wire maintains contact with faces, or edges, or vertices?
5. 1-1 cutting - can the object be cut out in such a way that each point of its faces is touched exactly once?
6. cutting-path description - can the ends of the wire move along any curve, or piece-wise linear curves only?

In this paper we will study the wire cutting problem for convex and non-convex polyhedral sets with face-wise and polyhedron-wise continuity of the cutting path. A problem of feasibility of cutting that relaxes the face-wise continuity requirement is presented in a separate paper.

Clearly, from a manufacturing point of view we need not only to answer the feasibility problem but to construct a description of the cutting path for the wire (via moving its endpoints) and develop a framework for constructing and describing cutting paths. It is possible to see that there are two major obstacles to this type of cutting. The first difficulty may be posed by the connectivity structure (for example a cube with a hole). The second arises from complex relations between faces of the polyhedral set. This issue is well illustrated with the following figures (we call them table 1 and table 2).



While the left object (table 1) can be cut out, the second cannot. Because of the square (cross section) shape of the legs and the positioning of the cube attached to the bottom of the table, portions of the legs cannot be removed from the second table as the wire would need to slice through the cube. When the legs have triangular cross sections (table 1), cutting out is possible; note that the bottom of the table requires five separate cuts (face-wise discontinuity).

Let us formalize our problem to be able to study it with computational geometry tools. Let P be an open polyhedral set (later simply called a polyhedron) whose closure has a total of n faces, edges and vertices. For technical reasons it is convenient to consider open polyhedrons rather than closed ones. (Note that the wire is peeling off the boundary of the object.)

We say that line l in R^3 cuts P if it has a nonempty intersection with the boundary of P. We say that P can be cut out if there is a path $p(l)$ of lines such that each l in this path cuts P and the union of the intersections with the closure of P is equal to the boundary of the closure of P. Such a path can be described with a pair of functions representing the location (in some coordinate system) of two selected points of the line (wire) as a function of time. In a hot wire cutter it can be described as the motion of the ends of the wire attached to the frame.

Furthermore, we say that a face f of the boundary of P is cut with face-continuity if there is a subpath of $p'(l)$ of $p(l)$ that cuts out f. That is, the union of intersections of lines in $p'(l)$ with the boundary of the closure of P equals f. Our hot wire cutting problem can be now defined as:

Given an open polyhedral set P find a cutting path $p(l)$ that preserves face-wise continuity, if it exists.

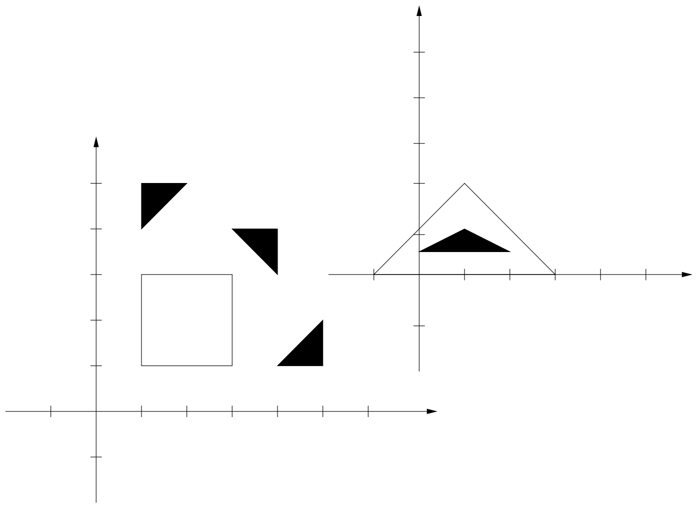
The concepts of (cutting) paths, subpaths and continuity can be formalized further by introducing a natural topology in the space of lines represented with Plücker coordinates, see [CEGS94, Pel97].

2 Wire cutting a face

Due to the space limitations we will present only results related to 1-1 face-wise continuous cutting (rather than a more general face-continuous cutting).

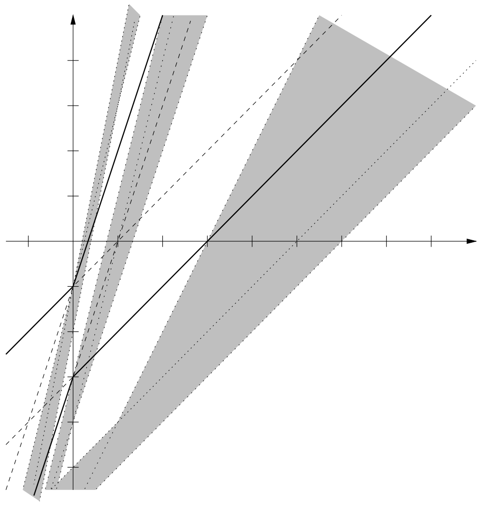
Let f be a face in the closure of P. Let $plane(f)$ be the supporting plane for f. Consider the intersection $plane(f) \cap clocure(P)$. This intersection is a set of polygons (possibly with holes) that include f and cross-sections $f_1, ..., f_k$ of $closure(P)$ with $plane(f)$. These polygons $f_1, ..., f_k$ are potential obstacles for the wire cutter to cut f. We notice immediately that the wire intersects f_i if and only if it intersects $conv(f_i)$. So in the further discussion we can assume that all of $f_1, ..., f_k$ are convex polygons. For each face f its obstacles can be found in $O(n)$ time.

We are interested in the following f-stabber problem: find all lines that intersect f without intersecting the interior of f_i, $i = 1, ..., k$. Figure below shows two instances of the problem (each in their own coordinate system). In this picture the unfilled objects corresponds to f and the filled ones correspond to the penetration of P.



It is easy to inspect that in the both cases there exists an f-stabber. In the first case the stabber has three disjoint components. In the hot wire cutting interpretation both (unshaded) polygons can be cut out. In the first case each of two connected components include a face-wide continuity path (moreover, there are 1-1 cutting paths). In the second case there is also a face-wide continuous path (but not 1-1 cutting).

This problem is a sort of traversal problem, and can be solved by using the dual projection [EMP+82]. Points are mapped into lines and vice versa using the following transformation: $(a, b) \rightarrow y = ax - b$. It is well known that segments are mapped into double wedges. In particular, the dual projection of a polygon is the union of the double wedges of the polygon edges. The combinatorial complexity of the stabbing region (the locus of the stabbers) of a set of m convex polygons with n vertices is $O(n\alpha(m))$, and is $O(n)$ if the polygons are disjoint (see [SA95]). Lines that stab f_i correspond in the dual space to points in the interior (since we stab open polygons) of this union. The set of lines that stab (cut) f without penetrating P is (in the dual space) a set of points in the difference of the dual image of f and the union of dual images of f_i. Note that this difference $D(f, f_1, ..., f_k)$ contains the boundary as we want to stab f without stabbing the interiors of f_i.



In the following f-stabber will be used in two meanings; the f-stabber and its dual image. The above figure illustrates the dual image of the first instance from the previous figure where the square corresponds to f and the filled triangles correspond to the penetration of P. The set of stabbing lines for f has (in the dual space) three connected components. Recall that f-stabber is defined as a collection of lines (in fact a path of lines) such that each point of f has a common point with some line in this collection. Even if f has an f stabber with respect to $f_1, ..., f_k$ such a stabber can be located in more than one of the connected components of $D(f, f_1, ..., f_k)$. In our example two of the components contain its own f-stabber.

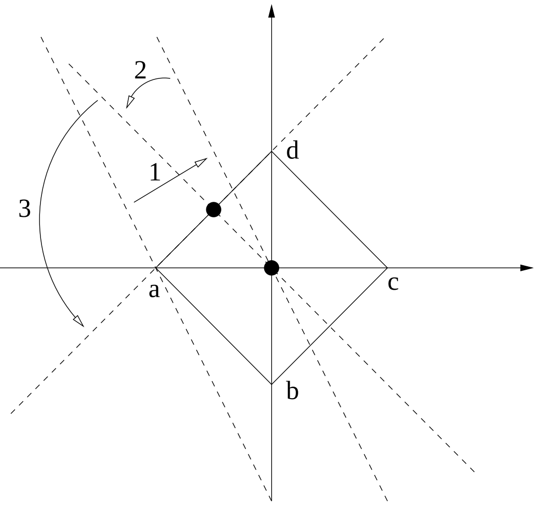
$D(f, f_1, ..., f_k)$ is the diference of the stabbing region (in the dual space) for f and the union of the stabbing regions for a set of polygons with a total of $O(n)$ vertices. It can be constructed and presented in a doubly connected edge list in $O(n \log n)$ time.

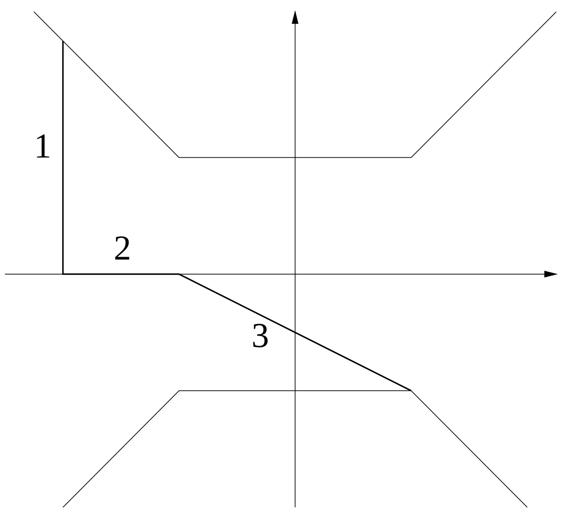
An algorithmic question here is: How can we decide if there is a f-stabber that contains a cutting path with face-wise continuity?

Before stating the algorithm let's note a few simple properties of the stabbing region $D(f, f_1, ..., f_k)$. A technical note: to allow for vertical lines, that are not included in a dual transformation, we consider arrangements of double wedges in the projective space by connecting infinite regions.

LEMMA **2.1** *(1) A vertical segment in $D(f, f_1, ..., f_k)$ corresponds to a family of parallel lines that intersect f. (2) A nonvertical segment corresponds to a pencil (a double wedge) of lines intersecting f. The center of this wedge is in f if and only if the line supporting this segment is contained in $D(f, f_1, ..., f_k)$.*

The lemma is illustrated in the following two figures. The first shows a square and a cutting sequence that involves a rotation about a point inside the square (and hence is not 1-1). The second shows the corresponding path in the dual space.





The lemma stems directly from the definition of the dual transformation and leads to the following result.

Let $L(f)$ and $U(f)$ be the lower and upper boundaries of the stabbing region (in the dual space) for f.

THEOREM **2.1** *There is a 1-1 cutting path for f with the face-wide continuity that consists of a sequence of translations and rotations of the cutting line if and only if there is a polyline contained in one component of the f-stabber which connects points in $L(f)$ and $U(f)$ and intersects each line in $D(f, f_1, ..., f_k)$ at most once.*

To comment on the last statement, a polyline that crosses a line of $D(f, f_1, ..., f_k)$ more than once corresponds to a cutting path that touches certain point of f (one whose dual is that line) more than once, violating the 1-1 requirement.

The above result can be extended to cutting paths that consists of superpositions of rotations and translations; such cutting paths correspond to trajectories in a connected component of the f-stabber that are not necessarily polylines.

Given a dual representation of the f-stabber, the lemma allows us to construct face-wide continuous cutting paths for f as follows. Scan the connected components of the f-stabber to identify if they contain some edges e_l or vertices v_l of $L(f)$ and e_u, v_u of $U(f)$. Let $v(e_l)$ be the vertex of f in the primary space that maps to e_l in $L(f)$. Similarly, denote the correspondence of v_l and $e(v_l)$, e_u and $v(e_u)$, v_u and $e(v_u)$. Each segment connecting v_l and v_u (note that they are always mutually visible) corresponds to the cutting path that cuts from $e(v_l)$ to $e(v_u)$ in the sense that the initial position of the cutting line contains $e(v_l)$ and the final position contains $e(v_u)$. Clearly, such a segment intersects each line at most once. Similarly, segments that connect points in e_l and e_u correspond to cutting paths that start at the respective vertices of f. Depending on the orientation of the segment, the cut will be achieved either with translation or with rotation. At this step we can also identify all pairs of edge-vertex, edge-edge and vertex-vertex continuous cuts for f. In fact, there is no need to list all the pairs. Instead it suffices to know edges and vertices in the respective parts of $U(f)$ and $L(f)$; all pairs of them are mutually visible (they have a butterfly shape) and as such can be connected with a line segment. This observation will be useful in creating a graph of feasible transitions from a face to a face of the polyhedral set.

Again, we are working in the projective space to handle vertical lines correctly.

Summarizing the results of this section we can state

THEOREM **2.2** *For each face f of P it is possible to construct 1-1 face-wise continuous paths in time $O(n \log n)$.*

Consequently, the cost of verifying the 1-1 face-wise continuity for all the faces of P is $O(n^2 \log n)$.

2.1 A graph of transitions

The last stage of the algorithm is to find if the polyhedral set P can be cut out with the polyhedral-wise continuity while maintaining other constraints.

Graph $G(P)$ of feasible transitions between faces of P depends on these constraints. For example, $G(P)$ for 1-1 cutting with edge to edge cutting can be constructed as follows. For each face f consider all the connected components that contain a 1-1 cutting path directed from an edge to an edge. Nodes of $G(P)$ correspond to such components. Nodes corresponding to components for different faces sharing an edge e in P are linked if and only if each of them has a path ending at e. Note that the graph is correctly defined. If e_1 and e_2 can end 1-1 paths for f then there is a 1-1 cutting path for f that starts at e_1 and ends at e_2. The links can be found as a byproduct of the algorithm from the previous section. Moreover, since the connected components of the f-stabber are monotonic with respect to x-axis, the size of $G(P)$ is $O(n)$.

THEOREM **2.3** *Transition graph $G(P)$ that contains information about 1-1 face-wise continuous cutting can be constructed in $O(n^2 \log n)$ time. Its size is $O(n)$.*

A schedule for 1-1, edge-to-edge cutting with face and polyhedral continuity exists if an only if $G(P)$ is connected and has a Hamiltonian cycle (we want to start and end at the same position). This creates difficulties as

THEOREM **2.4** *Finding a Hamiltonian cycle in $G(P)$ graphs is an NP-complete problem.*

It holds true even if P is limited to polytops. In this case $G(P)$ is the dual of P. Consider P a dual of a 3-polytope. Steinitz's theorem implies that a graph is the graph of a 3-polytope if and only if it is planar and 3-connected ([BL93]). When combined with results from [GT76] it gives the above result.

3 Conclusions

Hot wire cutting is a natural application of numerous results regarding sets of lines in two and three dimensional space.

We presented a solution to one of several algorithmic questions that arise in hot wire cutting of polyhedral sets. The nature of the face-wise continuity question allowed us to reduce the problem to a two dimensional f-stabber problem, and then to constructing a transition graph. Let us note that when the face-wise continuity is relaxed the methods start engaging sets of lines in 3-dim.

Only Hamiltonian paths in the transition graph allow paying a unique visit to each face of the polyhedral set and they are in general difficult to find. Using a spanning tree and doubling each edges of the tree, we have a reasonably good path. It crosses edges of f at most twice. To avoid multiple cuts of faces we can keep a track of already visited ones and instead of re-cutting them we can rotate the line around the boundary of f to reach the edge adjacent to the face that is next in the schedule.

Some of the methods used in the paper can be applied to a related problem of cutting with a laser beam. There are differences; geometrically a laser beam is a ray, technologically, it can be switched off and repositioned.

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