Every Polygon Can Be Untangled

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Main Theorem. We consider a simple polygon in the plane as a system of rigid bars with movable joints at the vertices (a *linkage* or *framework*). We show that such a polygon can always be continuously reconfigured into convex position while avoiding self-intersection. Similarly, an *open* polygonal path (chain) can always be straightened. During this motion, whereas the lengths of the bars remain fixed, all other pairwise distances increase.

1.1. History and Applications; Related Work. This type of problem has been considered in discrete and computational geometry since an early work of Erdős [5], in knot theory, and in molecular biology and polymer physics. Its applications include wire bending, hydraulic tube folding, and the study of macromolecule folding.

The present problem was posed by Joseph Mitchell in 1992, and by Lenhart and Whitesides in [6]. So far, only partial results for special classes of polygons have been obtained. It has also been known that every planar polygon can be convexified by reconfiguring through three-space [1, 7]. However, some three-dimensional unknotted polygons cannot be untangled [1, 3]. For *trees* in the plane it is known that they can have configuration spaces with exponentially many connected components [2]. For more on the history of the problem, see [8].

1.2. Overview of the Proof. We first prove the existence of a *local* infinitesimal motion, using the nonexistence of certain stresses, a notion dual to infinitesimal motions for the underlying framework. The analysis of these stresses use a lifting theorem that was known to J. C. Maxwell and L. Cremona in the nineteenth century from the theory of rigidity. Finally, we will put the local motions together to arrive at the desired global solution.

1.3. Notation. For brevity, we only consider polygons; the proof works also for an open polygonal chain. (In fact, this case can also be reduced to the case of a closed polygon.) We denote the polygon by P and the positions of its vertices by $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n)$. A *motion* of the polygon is a continuous function $\mathbf{p}(t) = (\mathbf{p}_1(t), \ldots, \mathbf{p}_n(t))$, for $0 \le t \le T$, where $\mathbf{p}(0) = \mathbf{p}$ and $\|\mathbf{p}_i(t) - \mathbf{p}_j(t)\|$ is constant for each edge $\{i, j\}$.

We can assume that P has no straight angles, because the two bars incident to a straight angle could be coalesced it into a single bar. Moreover, we assume throughout that P is nonconvex, because otherwise there is nothing to prove.

2. Excluding Self-Intersections. We insist that no distance between any two vertices decreases in our motion. The following easy lemma says that this is enough to ensure that the motion creates no self-intersection.

Lemma 1 *If a motion causes a self-intersection, then some pair of vertices must be closer than in the initial configuration.*

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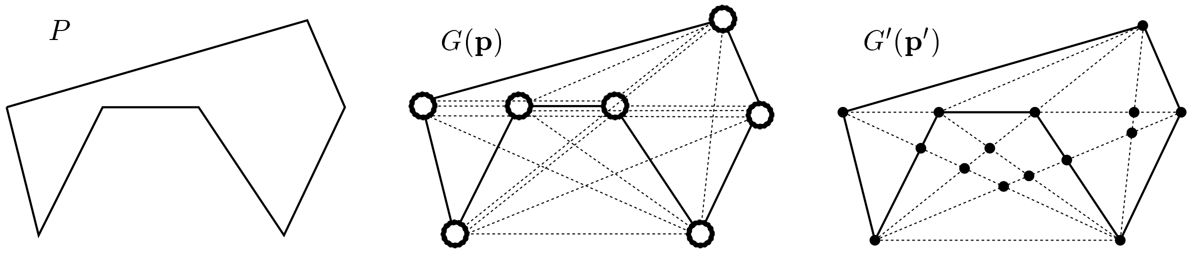

Figure 1: Construction of the frameworks $G(\mathbf{p})$ and $G'(\mathbf{p}')$. Solid lines denote bars, and dashed lines denote struts.

3. A Local Motion. We prove that there is an infinitesimal motion that increases all distances. We apply the theory of stresses in *tensegrity frameworks*, where there are *struts* that are permitted to increase in length, or stay the same length, as well as *bars* whose length remains fixed. We define the set B of bars as the set of polygon edges. The set S of struts consists of all vertex pairs which are not in B. The resulting framework is denoted by $G(\mathbf{p})$.

The desired distance-increasing property can be written as follows:

$$\begin{aligned} \tfrac{d}{dt}\|\mathbf{p}_j(t) - \mathbf{p}_i(t)\| &= 0 \text{ for } \{i,j\} \in B, \\ \tfrac{d}{dt}\|\mathbf{p}_j(t) - \mathbf{p}_i(t)\| &> 0 \text{ for } \{i,j\} \in S. \end{aligned}$$

Denoting the velocity vectors by $\mathbf{v}_i = \mathbf{v}_i(t) = \frac{d}{dt}\mathbf{p}_i(t)$, these conditions are equivalent to

$$\begin{aligned} (\mathbf{v}_j - \mathbf{v}_i) \cdot (\mathbf{p}_j - \mathbf{p}_i) &= 0 \text{ for } \{i,j\} \in B, \\ (\mathbf{v}_j - \mathbf{v}_i) \cdot (\mathbf{p}_j - \mathbf{p}_i) &> 0 \text{ for } \{i,j\} \in S. \end{aligned} \tag{1}$$

Theorem 1 *If the polygon is not convex, it has an infinitesimal motion satisfying* (1).

3.1. Stresses. The system (1) is a linear feasibility problem that is common for tensegrity frameworks. It is helpful to restate it in terms of a dual problem. A *stress* $\omega = (\ldots, \omega_{ij}, \ldots)$ of the framework $G(\mathbf{p})$ is an assignment of a scalar $\omega_{ij} = \omega_{ji}$ to each bar or strut $\{i,j\}$ such that the following equilibrium equation holds for each vertex i of G:

$$\sum_{j:\{i,j\}\in B\cup S} \omega_{ij}(\mathbf{p}_j - \mathbf{p}_i) = 0$$

The stress is *proper* if $\omega_{ij} \geq 0$ for all struts $\{i,j\}$. Theorem 1 can be reformulated as follows.

Theorem 2 *The framework $G(\mathbf{p})$ has only the zero proper stress.*

The equivalence of this statement to Theorem 1 follows from the Farkas lemma (or, if you prefer, linear programming duality); see [10, Theorem 4.3] for a similar result.

It might be more appealing to reverse the signs of all ω_{ij}'s and think of *cables* instead of struts, which transmit forces only as tensions. Theorem 2 then implies that a nonconvex polygon cannot be held in shape by cables (or strings, or rubber bands) between vertices.

3.2. Making the Framework Planar. We transform the framework $G(\mathbf{p})$ into an equivalent *planar* framework $G'(\mathbf{p}')$, see Figure 1: we introduce new vertices at all intersection points between edges of $G(\mathbf{p})$, and subdivide the edges accordingly. Any resulting multiple edges are merged. The framework $G'(\mathbf{p}')$ has bars wherever $G(\mathbf{p})$ has bars. All other edges of G' are struts.

Lemma 2 *If $G(\mathbf{p})$ has a nonzero proper stress ω, then $G'(\mathbf{p}')$ has a nonzero proper stress ω'.*

Proof: During the modifications to $G(\mathbf{p})$ that made $G'(\mathbf{p}')$, we define ω' as follows. When we subdivide an edge $\{i, j\}$ with stress ω_{ij}, each edge $\{k, l\}$ of the subdivision gets the stress $\omega'_{kl} := \omega_{ij}\|p_i - p_j\|/\|p_k - p_l\|$. When merging several edges, we add the corresponding stresses. It is easy to verify the equilibrium conditions. One can also check that positive and negative stresses do not cancel completely during the merging process, using the fact that no strut of $G(\mathbf{p})$ is completely covered by bars. □

3.3. The Maxwell-Cremona Correspondence; Polyhedral Terrains. To prove that $G'(\mathbf{p}')$ has no nontrivial proper stress, we employ the Maxwell-Cremona correspondence between stresses in planar frameworks and three-dimensional polyhedral terrains that project onto these frameworks: A *polyhedral terrain* Γ comes from lifting a planar framework into three dimensions—that is, assigning a z coordinate to each vertex in the framework—such that each face bounded by edges of the framework (including the exterior face) remains planar. Γ is the graph of a piecewise linear continuous function of two variables.

Consider an edge $\{i, j\}$ in a planar framework, separating faces F and F'. Let $z = \mathbf{a} \cdot \mathbf{p} + b$ and $z = \mathbf{a}' \cdot \mathbf{p} + b'$ be the two linear functions specifying Γ on F and F'. By a straightforward calculation, the vector $\mathbf{a}' - \mathbf{a}$ must be perpendicular to the edge $\{i, j\}$:

$$\mathbf{a}' - \mathbf{a} = \omega_{ij}\mathbf{e}_{ij}^{\perp}, \tag{2}$$

where $\mathbf{e}_{ij}^{\perp}$ is a vector of the same length as the vector $\mathbf{p}_j - \mathbf{p}_i$, perpendicular to it, and pointing from F to F'. We call the edge $\{i, j\}$ a *valley* if $\omega_{ij} > 0$, a *mountain* if $\omega_{ij} < 0$, and *flat* if $\omega_{ij} = 0$.

Theorem 3 (Maxwell-Cremona Theorem) [4, 11] *(i) For every three-dimensional polyhedral terrain Γ that projects to a planar framework $G(\mathbf{p})$, the relations* (2) *define a stress ω on $G(\mathbf{p})$.*

(ii) For every stress ω in a planar framework $G(\mathbf{p})$, $G(\mathbf{p})$ can be lifted into a polyhedral terrain Γ such that (2) *holds for all edges. Furthermore, Γ is unique up to addition of affine-linear functions.*

Zero stresses corresponds to trivial terrains in which all faces are coplanar. Therefore, we need to show that there is no nontrivial polyhedral terrain over the framework $G'(\mathbf{p}')$ with the property that all struts are lifted to valleys or flat edges.

By adding a suitable affine-linear function, we may assume that exterior face of Γ is the x-y-plane. Now let $M \neq \emptyset$ denote the region in the plane where Γ attains its maximum. For contradiction, let us assume that M does not fill the whole plane, and hence the boundary ∂M is nonempty. Consider first the case of an isolated point v in ∂M. Intersect Γ with a horizontal plane just below v. The result is a star-shaped polygon, with convex vertices corresponding to mountains and reflex vertices corresponding to valleys emanating from v. But a polygon must have at least three convex vertices, whereas v can be incident to only two bars, a contradiction.

The case when v is a degree-1 vertex of ∂M leads to a similar contradiction. Now, every edge in ∂M must be a mountain, and therefore form a bar, and therefore be part of an edge of the given polygon P. It follows than ∂M must coincide with P. The only possibility is therefore that M contains the exterior face of P but does not contain the interior of P. Then, applying the above argument (cutting Γ just below M) to some reflex vertex of ∂M, one derives a similar contradiction as above. This completes the proof of Theorems 2 and 1.

4. The Global Motion. Now we prove our main claim, that every polygon can be "untangled". We define for each nonconvex configuration $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n)$ a velocity vector $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_n) =: f(\mathbf{p})$ as the solution of the convex quadratic programming problem:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{n} \|\mathbf{v}_i\|^2 \\ \text{subject to} & (\mathbf{v}_j - \mathbf{v}_i) \cdot (\mathbf{p}_j - \mathbf{p}_i) = 0 \text{ for } \{i, j\} \in B, \\ & (\mathbf{v}_j - \mathbf{v}_i) \cdot (\mathbf{p}_j - \mathbf{p}_i) \geq 1 \text{ for } \{i, j\} \in S, \end{array} \tag{3}$$

Since the objective function is strictly convex, $\mathbf{v}$ is unique. The function $f(\mathbf{p})$ is defined on an open set $U \subset \mathbf{R}^{2n}$ that is characterized by the conditions that no angles are $0°$ or $180°$, no vertex touches a link, and the polygon is nonconvex.

We will show that the differential equation

$$\frac{d}{dt}\mathbf{p}(t) = f(\mathbf{p}(t)),\ \mathbf{p}(0) = \mathbf{p}_0 \tag{4}$$

provides the desired motion. The stability theory of parametric quadratic programming [9, Theorem 2; see in particular p. 140] guarantees that f is locally Lipschitz-continuous in U, provided that the set of primal-dual pairs $(\mathbf{v}, u)$ fulfilling the Karush-Kuhn-Tucker necessary optimality conditions corresponding to (3) is bounded. This can be shown using strict convexity of the objective function in connection with Theorem 2 (with $u_{ij} = \omega_{ij}$).

Local Lipschitz continuity of f in U is sufficient to ensure that the initial value problem (4) has a (unique) *maximal solution* $\mathbf{p}(t)$, $0 \le t \le T$, that extends to the boundary of U. At the boundary of U, an angle of P becomes straight. We coalesce two neighboring edges and proceed with a polygon with one edge less, until P eventually becomes convex. This concludes the proof.

5. Extensions and Related Problems. Our result holds equally for closed and for open polygons (polygonal chains). In the full version of our paper, which is under preparation, we also treat the case of several open and closed polygons, as well as certain polygons in three dimensions.

A natural question concerns the configuration space of a polygon. As an easy consequence of our result, we obtain that the configuration space of an open polygonal chain is connected, and the configuration space of a polygon has two connected components, corresponding to the two orientations. Moreover, we can show that the configuration space modulo congruences (including orientation reversing ones) is contractible.

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