On computing tangent visibility graphs

Short extended abstract

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Abstract

We present a new method to implement in constant amortized time the flip operation of the so-called 'Greedy Flip Algorithm' (GFA), an optimal algorithm to compute the tangent visibility graph of a collection of pairwise disjoint bounded convex sets (*disks* for short) of constant complexity. The method uses only primitive data structures and only the primitive predicate which states that the angle of a first bitangent is less than the angle of a second bitangent, both bitangents being tangent to the same disk.

1 Introduction

Let $\mathcal{O} = \{O_i\}$ be a finite family of $n \geq 2$ pairwise disjoint bounded closed convex subsets of the Euclidean plane with nonempty interiors and smooth boundaries (*disks* for short), and recall that a *pseudotriangle* T is a simply connected subset of the plane, such that (1) its boundary ∂T consists of three convex curves, the *sides* of T, that share a tangent at their common endpoint, (2) T is contained in the triangle formed by the three endpoints of these convex curves. These three endpoints are called the *cusps* of T. A *(free) bitangent* is a closed line segment whose supporting line is tangent to two disks at its endpoints (and whose interior lies in *free space*, i.e., the complement of the union of the disks). The number of free bitangents is denoted by k. The endpoints of the free bitangents subdivide the boundaries of the disks into connected pieces, called *arcs.* The *tangent visibility graph* is the graph whose vertices are the endpoints of the free bitangents and whose edges are the free bitangents and the arcs. A *pseudotriangulation* is a maximal subset of pairwise disjoint free bitangents. It is known that a pseudotriangulation contains $3n-3$ bitangents, that subdivide free space into pseudotriangles. A crucial property is that given two disjoint pseudotriangles (of some pseudotriangulation) there exists a unique bitangent whose supporting line is tangent to the given pseudotriangles at its endpoints. To *flip* a bitangent, say b, in a given pseudotriangulation means to replace b by the bitangent whose supporting line is tangent at its endpoints to the pseudotriangles of the given pseudotriangulation incident upon b. A *signed* disk is a disk with an orientation of its boundary.

The goal of the paper is to show that the tangent visibility graph of the family of n disks is computable in optimal $O(k + n \log n)$ time and linear working space using only *primitive* data structures, — that is arrays and/or linked structures (collections of nodes interconnected by pointers) to represent maps, sets, and lists [2]—, and the primitive predicate $\chi_1(bb')$, which states that the directed bitangents b and b' are tangent to the same signed disk and $\Theta(b) < \Theta(b')$—here $\Theta(b)$ is the positive (counterclockwise) angle over which we have to rotate a unit vector parallel to the horizontal positive direction in order to obtain a vector parallel to b. The reflexive and transitive closure of the χ_1 relations defines a partial order $\prec$ on the

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set B of free directed bitangents. We denote by $\prec_*$ the dual partial order which is defined on the same set of bitangents by $b \prec_* b'$ if and only if $b' \prec b$. Any object X defined with respect to the partial order $\prec$ has a dual version defined with respect to the dual partial order $\prec_*$. This dual version is denoted X_*.

Our starting point is the so-called 'Greedy Flip Algorithm' (GFA) which is an optimal time and linear working space algorithm to compute the tangent visibility graph (see [1]). The basic idea of the GFA is to compute a linear extension of the poset $(B, \prec)$. Let $I(\theta)$ be the filter of bitangents with angle $\geq \theta$. Each filter $I \supset I(\pi)$ is associated with its so-called greedy pseudotriangulation $G(I) = \{b_1, b_2, \ldots, b_{3n-3}\} \subset I$ defined as follows: (1) b_1 is minimal in I; (2) b_{i+1} is minimal in the subset of bitangents in I (interior) crossing none of the bitangents $b_1, b_2, \ldots, b_i$. Since crossing bitangents are comparable (see [1]), $G(I)$ is well-defined and is a supset of $\min_\prec I$.

The GFA maintains the greedy pseudotriangulation $G(I)$ while I ranges over a maximal chain of filters in the interval $[I(0), I(\pi)]$. The basic operation that updates the pseudotriangulation is the flip of a free bitangent, minimal in the filter. The correctness of the GFA is a consequence of the Flip Property.

Theorem 1 (Flip Property [1]). *Let $b \in \min_\prec I$. Then*

$$G(I \setminus b) \setminus G(I) = \{\phi(b)\}$$

where $\phi(b)$ the bitangent that leaves $R(b)$ and enters $L(b)$ — here $R(b)(L(b))$ is the pseudotriangle of $G(I)$ incident upon b and—locally—to the right (left) of b. □

To implement efficiently the flip operation, the GFA uses *advanced* data structures — namely splittable queues —, the primitive predicate $\chi_2(bb')$, which states that the directed bitangents b and b', not necessarily tangent to the same signed disk, satisfy $\Theta(b) < \Theta(b')$, and the primitive predicate $\chi_3(bb')$, which states that the headpoint of b is on the left side of the supporting line of b' (see [1]). In this paper we show that the flip operation of the GFA can be implemented efficiently using only primitive data structures and the weaker primitive predicate χ_1.

The paper is organized as follows. In the next section we describe the new method to implement the flip operation. This is done into two steps: first we describe a method (called FLIP1) that uses only primitive data structures but still the predicate χ_2, then we show how to modify the FLIP1 to use only the χ_1 predicate. Technical developments are postponed to Section 3; no proofs are given in this short extended abstract.

2 The flip operation

Using only the predicate χ_2. As said in [1]: "Conceptually the flipping can be done by walking — in the positive direction — along the boundaries of the pseudotriangles L (left) and R (right) incident upon the flipped bitangent b, with one leg in every pseudotriangle, such that at any moment the tangent lines at the points underneath our left and right legs are parallel. We keep walking until these tangent lines coincide. At that point we have found $\phi(b)$". The following piece of code is a discrete version of this continuous parallel walk.

Function WALK$(r, l$: ARC$)$: BITANGENT
1 $\quad \phi \leftarrow \mathrm{bit}(r, l)$;
2 $\quad$ **if** (ϕ leaves r and enters l) **return**(ϕ);
3 $\quad$ **if** $(r^+ l^+)$ WALK (r', l) **else** WALK (r, l');

where, in line 1, $\mathrm{bit}(r, l)$ is the (possibly non free) bitangent leaving the signed disk supporting the arc r and entering the signed disk supporting the arc l. r' is the successor of the arc r on the boundary of $R(b)$. r^+ is the bitangent that follows the arc r on the boundary of $R(b)$. Note that the predicate χ_2 is used at line 3 while the test of line 2 involves only the predicate χ_1.

The key question is where to start the walk, say on R (the story is completely similar for L). Since $\phi(b)$ leaves either the second or the third side of R a lazy and simple solution is to start the walk at the arc that follows x_R the first cusp point of R. However there are examples where this choice is too costly.

The right solution is to start the walk at the arc, denoted r_{good}, that follows the *good* cusp point : (1) If b and b_R (the successor of b in R) are separed by x_R then x_R is declared to be good, (2) If b and b_R are not separated by x_R let $\phi_R(b)$ be the bitangent that leaves $R(b_R)$ and enters $L_*(b_R)$—the dual version of $L(b_R)$—, if $\phi_R(b)$ is right-xx[1] then x_R is declared good otherwise it is y_R the second cusp point of R which is declared good[2]. A crucial property at this point is that $\phi_R(b)$ is also the bitangent that leaves $R(b)$ and enters $T_{\text{forw}}(b)$ the pseudotriangle of the dual pseudotriangulation $G_*(I_*(0))$ on which b enters.

The following function, called with the parameters $(r_{\min}, l_{\min}, l_{\min})$ where $r_{\min}$ is the minimal arc of the second side of $R(b)$ and $l_{\min}$ is the arc of $T_{\text{forw}}(b)$ that b enters, returns $\phi_R(b)$ in constant amortized time. Note that only the χ_1 predicate is used. The correctness is based on properties of greedy pseudotriangles.

Function PHIR(r, l, g : ARC) : BITANGENT

- $\phi \leftarrow \text{bit}(r, l)$; $\phi' \leftarrow \text{bit}(r_{max}, g)$;
- **if** ϕ' leaves $r_{\max}$ and enters g **return**(ϕ');
- **if** ($r^+\phi$) PHIR(r', l, g');
  - **elseif** ($l^+\phi$) PHIR(r, l', g');
  - **else return**(ϕ);

where l' and g' are the successors of l and g on the boundary of $T_{\text{forw}}(b)$, and where r' is the successor of r on the boundary of $R(b)$. r_{max} is the arc, if any, that follows the second cusp point of $R(b)$.

We sum up the discussion in the following piece of code

[1] A right-xx (left-xx) bitangent is a bitangent that leaves a counterclockwise (clockwise) oriented disk.

[2] An alternative is to use, instead of $\phi_R(b)$, the ray $s(b)$ that leaves $R(b)$ and enters the headpoint of b. $s(b)$ is left-xx iff. $\chi_3(bb')$ where b' is the bitangent that defines the second cusp point of R. The drawback of this alternative, from our point of view, is that it requires to use the predicate χ_3. Note that this predicate is easy to compute in the case of polygonal obstacles since the endpoints of the bitangents are vertices of the input polygons — so it might be the method of choice in that case. Things are less clear for general disks.

Function FLIP1(b : BITANGENT) : BITANGENT

- compute r_{good} and l_{good} using PHIR and PHIL
- WALK($r_{\text{good}}, l_{\text{good}}$)

Theorem 2. *The time complexity of* FLIP1*(b) is constant amortized per flipped bitangent.* □

The key idea of the proof is to charge the cost of passing a bitangent $a \in R(b)$ on the bitangent, denoted $\phi(a, b)$, that leaves $R(a)$ and enters $L(b)$. It turns out that, except for $O(k)$ bitangents pairs (a, b), $\phi(a, b)$ is a well-defined free bitangent (Theorem 5) and the operator $\phi(a, b)$ is roughly speaking one-to-one (Theorem 4). The proof is based on mixed properties of the partial order $\prec$ and its dual $\prec_*$ (Theorem 3).

Using only the predicate χ_1. We only describe the method in the case where $\phi_L(b)$ and $\phi_R(b)$ are smaller than $\phi(b)$, and $\phi(b)$ is a right-left bitangent. The general case can be treated similarly. Let $t_2, t_3, \ldots$ be the increasing sequence of bitangents lying on the second side of $R(b)$ and greater than $t_1 := \phi_R(b)$. Similarly let $t'_2, t'_3, \ldots$ be the increasing sequence of bitangents lying on the second side of $L(b)$ and greater than $t'_1 := \phi_L(b)$. Let $l_{\min}$ ($r_{\min}$) be the arc that follows t'_1 (t_1) on the boundary of $L(b)$ ($R(b)$). The method is based on the following claims
(1-)$\phi(b) = \phi_R(t'_k)$ where k is the first index such that $\phi_R(t'_k)$ enters $L(b)$.
(2-)$\phi_R(t'_i)$ leaves $R(b)$ and enters $L_*(t'_{i+1})$.
Hence we compute successively the bitangents $\phi_R(t'_1), \phi_R(t'_2), \ldots$ until we detect t'_k with the condition stated in claim (1-) above. According to claim (2-) $\phi_R(t_i)$ is computable with (a slight variation of) the function PHIR since $R(b)$ is known at this step of the GFA— whereas $R(t'_{i+1})$ is not. However it is too costly to call PHIR at $r_{\min}$ for the walk on the boundary of $R(b)$. We need to determine a more favorable arc. The favorable arc r_{start} is the arc that follows the minimal bitangent t_i in the sequence $t_1, t_2, \ldots$ such that $\phi_L(t_i)$ is xx-left. $\phi_L(t_i)$ is computed as the bitangent that leaves $R_*(t_{i+1})$ and enters $L(b)$ by calling the function PHIL at $l_{\min}$ for the walk on the boundary of $L(b)$. We denote by FLIP2 this second flip method. It uses only the χ_1 predicate and its complexity constant amortized per flipped bitangent.

3 Technical developments

Duality and amortization theorems. We assume that the reader is familiar with the notion of visibility complex [1]. Recall that a free bitangent is incident to six 2-faces of the visibility complex. We denote by $\chi_{rl}(b)$ the 2-face whose minimal (with respect to $\prec$) bitangent is b. It is known that $\phi_{rl}(b)$ the maximal bitangent of $\chi_{rl}(b)$ is the bitangent $\phi(b)$. We denote by $\chi_{rr}(b)$ the 2-face incident to b defined by any ray in the neighbourhood of b that is parallel to b and on its right side; the maximal bitangent of $\chi_{rr}(b)$ is denoted $\phi_{rr}(b)$; similarly we introduce the notations $\chi_{ll}(b)$ and $\phi_{ll}(b)$. The left boundary $\partial_L(\sigma)$ of a bounded 2-face σ is the set of bitangents t in σ such that $\phi_{rr}(t)$ is the maximal bitangent of σ. Similarly we introduce the right boundary $\partial_R(\sigma)$ with respect to the ϕ_{ll} operator. Let b be a free bitangent and let $\sigma = \chi_{rr}(b)$. If $b = \max \partial_L(\sigma)$ then we set $\phi_R(b) = \phi_{rr}(b)$ otherwise we set $\phi_R(b)$ to be the bitangent that leaves $R(b_R)$ and enters $L_*(b_R)$ where b_R is the successor of b in $\partial_L(\sigma)$ (or equivalently the successor of b on the boundary of the right greedy pseudotriangle $R(b)$). Similarly we introduce a left version ϕ_L of ϕ_R.

Let $R'(b)$ (resp. $L'(b)$) be the set of bitangents a of the second side of $R(b)$ (resp. $L(b)$) such that
(1-) a precedes $\phi(b)$, i.e. $a \prec \phi(b)$,
(2-) $a \notin \chi_{rl}(b)$, and
(3-) $a \neq \phi_{rr}(b)$ (resp. $a \neq \phi_{ll}(b)$)

Theorem 3 (Duality Theorem). *We claim that $a \in R'(b)$ iff. $b \in L'_*(a)$* □

For $b \in B$ we set $Z_R(b)$ (resp. $Z_L(b)$) to be the set of bitangents that belong to $R'_*(b)$ (resp. $L'_*(b)$) or to $R'(b)$ (resp. $L'(b)$), or to the right (left) boundary of the face $\chi_{rr}(b)$ ($\chi_{ll}(b)$). Note that (1) $Z_R(b)$ and $Z_L(b)$ are disjoint and (2) the bitangents in $Z_R(b)$ ($Z_L(b)$) are linearly ordered with respect to $\prec$. For $a \in Z_R(b)$ (resp. $a \in Z_L(b)$) we define $\phi(a, b)$ to be the common bitangent (which is not necessarily free) to $R(a)$ and $L(b)$ (resp. to $L(a)$ and $R(b)$) that belongs to the interval $[a, \pi(a)]$. Let $\phi_*(.,.)$ be the dual version of $\phi(.,.)$. Finally let $\mathcal{G}_R$ be the bipartite graph $\bigcup_{b\in B} (Z_R(b) \cup \{b\}) \times \{b\} \subseteq B \times B$; note that $\mathcal{G}_R \supseteq \mathcal{D}$ where $\mathcal{D} = \{(b, b) \mid b \in B\})$ is the main diagonal of $B \times B$.

Theorem 4 (Amortization Theorem). *Let H be a subgraph of the bipartite graph $\mathcal{G}_R$ such that $H \supseteq \mathcal{D}$. We claim that*

$$|H| = O(|\phi(H)| + |\phi_*(H)|). \tag{1}$$

Complexity analysis of Flip1 **and** Flip2**.** It should be clear at this point that the complexity of the procedure Flip1 is up to an $O(k)$ additive factor proportional to the quantity $C_1 = \sum_b |R'(b)|$ where b ranges over the set of bitangents t such that $\phi_R(t)$ is right-xx. Similarly the complexity of Flip2 is up to an $O(k)$ additive factor proportional to the quantity $C_2 = \sum_b |W_R(b)|$ where b ranges over the set of bitangents t such that $\phi_R(t)$ is right-xx and where $W_R(b)$ is the set of bitangents $t \in Z_R(b)$ such that $\phi_L(t)$ is xx-left. The following theorem relates the type of $\phi_R(b)$ to the freeness of $\phi(a, b)$ and therefore allows us to apply the amortization theorem to conclude that C_1 and C_2 are in $O(k)$.

Theorem 5 (Freeness Theorem).

1. *Let $b, b' \in L'_*(a)$ with $b \prec b'$, and assume that $\phi_R(b)$ is right-xx. Then $\phi_R(b')$ is right-xx and $\phi(a, b')$ and $\phi_*(a, b')$ are free.*
2. *Let $b, b' \in Z_L(a) \cap Z_L(a')$ with $b \prec b'$ and $a \prec a'$, and assume $\phi_R(b)$ is right-xx and $\phi_L(a)$ is xx-left. Then $\phi_R(b')$ is right-xx, $\phi_L(a')$ is xx-left and $\phi(a', b')$ and $\phi_*(a', b')$ are free.* □

References

[1] M. Pocchiola and G. Vegter. Topologically sweeping visibility complexes via pseudo-triangulations. *Discrete Comput. Geom.*, 16:419–453, December 1996.

[2] R. E. Tarjan. *Data Structures and Network Algorithms*, volume 44 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics, Philadelphia, PA, 1983.