$\boldsymbol{\alpha}$-Sectors of Rational Varieties

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1 Introduction

Given two varieties V_1, V_2, the *bisector* of V_1 and V_2 consists of all the points that are equidistant from the two varieties. Given a set of varieties, portions of bisectors of pairs of varieties form the edges of the Voronoi diagram of the set. In this work we extend the bisector to a more general concept, so-called the α-*sector*. Instead of taking an equal distance from two varieties, the α-sector allows different relative distances from the two varieties. Even in the simple case of the α-sector of a point and a line, we can generate all different types of conics, depending on the value of $0 < \alpha < 1$. Exact α-sectors of rational varieties are nonrational even in the special cases where the bisectors are rational. We therefore propose the pseudo α-sectors that approximate exact α-sectors with rational varieties. Both exact and pseudo α-sectors reduce to bisectors for $\alpha = \frac{1}{2}$. The ability to continuously change α is a useful tool in many applications, e.g., for metamorphosis between two different freeform shapes.

2 Point-Line $\boldsymbol{\alpha}$-Sector in $\mathbb{R}^2$

We assume without loss of generality that the line is the Y-axis, that is, the parametric line $C(t) = (0, t)$, and that the point is $Q = (1, 0)$. We set $\alpha = 0$ for the line and $\alpha = 1$ for the point.

The α-sector $\mathcal{B} = (b_x, b_y)$ of the Y-axis and the point Q satisfies the line-orthogonality constraint

$$0 = \left\langle \mathcal{B} - C(t), \frac{dC(t)}{dt} \right\rangle = \langle (b_x, b_y) - (0, t), (0, 1) \rangle = b_y - t, \tag{1}$$

and the distance constraint

$$\alpha^2 \left((b_x - 1)^2 + b_y^2 \right) = (1 - \alpha)^2 (b_x^2 + (b_y - t)^2). \tag{2}$$

The solution of Equations (1)–(2) is the quadratic curve

$$\left(\frac{2\alpha - 1}{\alpha^2} \right) x^2 + y^2 - 2x + 1 = 0, \tag{3}$$

where (x, y) is used instead of (b_x, b_y).

Figure 1 shows the α-sectors of the line $(0, t)$ and the point $(1, 0)$ for various different values of α. When $\alpha < \frac{1}{2}$, the coefficients of x^2 and y^2 have opposite signs, and so the α-sector is a hyperbola. When $\alpha = \frac{1}{2}$, the coefficient of x^2 vanishes, and so the bisector is a parabola. When $\alpha > \frac{1}{2}$, the coefficients of x^2 and y^2 have the same sign, and so the α-sector is an ellipse.

We have shown above that the α-sector of a point and a line is a conic section—a hyperbola, a parabola, or an ellipse—depending on the value of α. In the full version of the paper we also show that every conic

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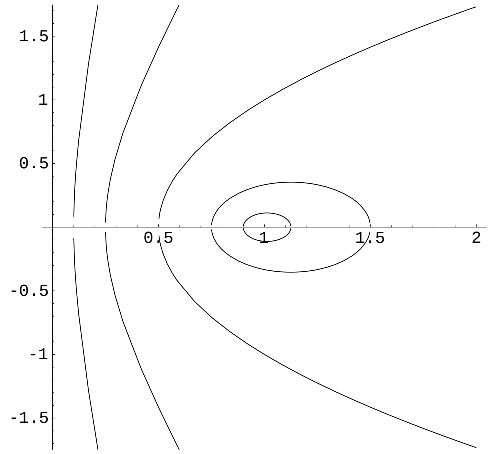


Figure 1: The α-sectors of the point $(1,\ 0)$ and the line $(0,\ t)$ for $\alpha = 0.10,\ 0.25,\ 0.50,\ 0.75,\ 0.90$.

section is an α-sector (for some value of $0 < \alpha < 1$) of some point and some line in the plane. We do that constructively and show how to compute the point, line, and α from the quadratic curve.

3 Point-Plane α-Sector in $\mathbb{R}^3$

A similar α-sector exists for a point and a plane in three dimensions. We assume without loss of generality that the plane is the YZ-plane, that is, it is the parametric plane $S(u, v) = (0, u, v)$, and that the point is $Q = (1, 0, 0)$. We set $\alpha = 0$ for the plane and $\alpha = 1$ for the point.

Let $\mathcal{B} = (b_x, b_y, b_z)$ be the α-sector of $S(u, v)$ and Q. As in the two-dimensional case we have the two plane-orthogonality constraints

$$0 = \left\langle \mathcal{B} - S(u,v), \frac{\partial S(u,v)}{\partial u} \right\rangle = \langle (b_x, b_y, b_z) - (0, u, v), (0, 1, 0) \rangle = b_y - u, \tag{4}$$

$$0 = \left\langle \mathcal{B} - S(u,v), \frac{\partial S(u,v)}{\partial v} \right\rangle = \langle (b_x, b_y, b_z) - (0, u, v), (0, 0, 1) \rangle = b_z - v, \tag{5}$$

and the distance constraint

$$\alpha^2 \left((b_x - 1)^2 + b_y^2 + b_z^2 \right) = (1 - \alpha)^2 (b_x^2 + (b_y - u)^2 + (b_z - v)^2). \tag{6}$$

The solution of Equations (4)–(6) is the quadratic surface

$$\left(\frac{2\alpha - 1}{\alpha^2} \right) x^2 + y^2 + z^2 - 2x + 1 = 0, \tag{7}$$

where (x, y, z) is used instead of (b_x, b_y, b_z). This is a hyperboloid for $0 < \alpha < \frac{1}{2}$, an elliptic paraboloid for $\alpha = \frac{1}{2}$, and an ellipsoid for $\frac{1}{2} < \alpha < 1$.

4 Line-Line α-Sector in $\mathbb{R}^3$

Yet another simple example is the α-sector of two lines $C_1(u) = (1, u, 0)$ and $C_2(v) = (0, 0, v)$. We set $\alpha = 0$ for $C_2(v)$ and $\alpha = 1$ for $C_1(u)$. Let $\mathcal{B} = (b_x, b_y, b_z)$ be the α-sector of $C_1(u)$ and $C_2(v)$. Here we have the two line-orthogonality constraints

$$0 = \left\langle \mathcal{B} - C_1(u), \frac{dC_1(u)}{du} \right\rangle = \langle (b_x, b_y, b_z) - (1, u, 0), (0, 1, 0) \rangle = b_y - u, \tag{8}$$

$$0 = \left\langle \mathcal{B} - C_2(v), \frac{dC_2(v)}{dv} \right\rangle = \langle (b_x, b_y, b_z) - (0, 0, v), (0, 0, 1) \rangle = b_z - v, \tag{9}$$

and the distance constraint

$$\alpha^2 \left((b_x - 1)^2 + (b_y - u)^2 + b_z^2 \right) = (1 - \alpha)^2 (b_x^2 + b_y^2 + (b_z - v)^2). \tag{10}$$

The solution of Equations (8)–(10) is the quadratic surface

$$\left(\frac{2\alpha - 1}{\alpha^2} \right) x^2 - \left(\frac{1 - \alpha}{\alpha} \right)^2 y^2 + z^2 - 2x + 1 = 0. \tag{11}$$

The $\frac{1}{2}$-sector (bisector) of $C_1(u)$ and $C_2(v)$ is thus the surface

$$y^2 - z^2 + 2x - 1 = 0,$$

whose parametric form is given as $(\frac{1-u^2+v^2}{2}, u, v)$. This repeats the result of [Elb, §2.2].

For $\alpha < \frac{1}{2}$, Equation (11) yields a hyperboloid of two sheets. For $\alpha = \frac{1}{2}$ it yields a hyperbolic paraboloid, and for $\alpha > \frac{1}{2}$ it yields a hyperboloid of one sheet, which reduces to a cylinder for $\alpha = 1$. However, the α-sector of two general parametric curves in $\mathbb{R}^3$ is in general a nonrational surface.

5 Pseudo α-Sector

We look for a linear constraint that can measure the relative distances to the two given varieties. While the L_2 norm is quadratic, we propose as a linear constraint the plane that is at $\alpha : (1 - \alpha)$ relative distances from the two examined points on the two varieties.

For example, for the pseudo α-sector of a curve $C(t)$ and a point Q in $\mathbb{R}^2$, we will impose the two linear constraints

$$\left\langle \mathcal{B}(t) - C(t), \frac{dC(t)}{dt} \right\rangle = 0, \tag{12}$$

$$\langle \mathcal{B}(t) - (\alpha Q + (1 - \alpha) C(t)), C(t) - Q \rangle = 0. \tag{13}$$

While Equation (12) is the regular orthogonality constraint, the constraint of Equation (13) ensures that the bisector is on the plane containing the point $\alpha Q + (1 - \alpha) C(t)$ and orthogonal to the vector $C(t) - Q$. If $C(t)$ has a rational representation, we can easily use Cramer's rule to obtain a rational representation for $\mathcal{B}(t) = (b_x(t), b_y(t))$.

Figure 2 shows three examples of planar pseudo α-sectors of: (i) a point and a line (Figure 2(a)), (ii) a point and a cubic curve (Figure 2(b)), and (iii) a point and a circle (Figure 2(c)).

The extension to $\mathbb{R}^3$ follows the same guidelines. The pseudo α-sector of two curves $C_1(u)$ and $C_2(v)$ in $\mathbb{R}^3$ imposes the three linear constraints

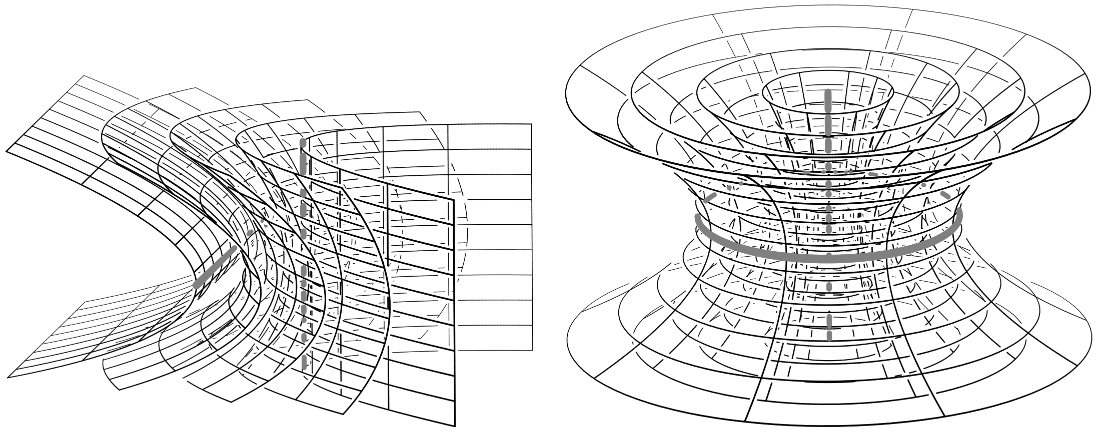
$$\left\langle \mathcal{B}(t) - C_1(u), \frac{dC_1(u)}{du} \right\rangle = 0, \tag{14}$$

$$\left\langle \mathcal{B}(t) - C_2(v), \frac{dC_2(v)}{dv} \right\rangle = 0, \tag{15}$$

$$\langle \mathcal{B}(t) - (\alpha C_1(u) + (1.0 - \alpha) C_2(v)), C_1(u) - C_2(v) \rangle = 0. \tag{16}$$

(a) (b) (c)

Figure 2: (a) The pseudo α-sectors of a point and a line in $\mathbb{R}^2$ for $\alpha = 0.10,\ 0.25,\ 0.50,\ 0.75,\ 0.90$. (Compare with Figure 1.) (b) The pseudo α-sectors of a point and a cubic curve in $\mathbb{R}^2$ for $\alpha = 0.2,\ 0.4,\ 0.6,\ 0.8,\ 1.0$. (c) The pseudo α-sectors of a point and a circle in $\mathbb{R}^2$ for $\alpha = 0.2,\ 0.4,\ 0.6,\ 0.8,\ 1.0$. The original curves and points are shown in gray.



(a) (b)

Figure 3: (a) The pseudo α-sectors of two lines in $\mathbb{R}^3$ for $\alpha = 0.0,\ 0.25,\ 0.5,\ 0.75,\ 1.0$. (b) The α-sectors of a line and a circle in $\mathbb{R}^3$ for $\alpha = 0.0,\ 0.25,\ 0.5,\ 0.75,\ 1.0$. The original curves are shown in gray.

Again, if $C_1(u)$ and $C_2(v)$ have rational representations, we can use Cramer's rule to obtain a rational representation for $\mathcal{B}(t)$. Figure 3 shows two such three-space pseudo α-sectors of: (i) two lines (Figure 3(a)) and (ii) a line and a circle (Figure 3(b)).

The pseudo α-sector is clearly different from the α-sector. They are identical only for $\alpha = \frac{1}{2}$, where they are also equivalent to the regular bisector. Note that unlike the regular α-sectors, the pseudo 0- and 1-sectors are different from the original varieties. This is because of the weaker distance constraint: points on the pseudo bisector do not satisfy the $\alpha : (1-\alpha)$ distance ratio; this property is imposed only on their projections on the lines joining the respective points on the varieties.

We finally mention two properties of the pseudo 0-sector of a point and a curve. (Here $\alpha = 0$ and $\alpha = 1$ correspond to the point and to the curve, respectively). Unlike the 0-sector of a point and a curve, the pseudo 0-sector does not identify with the point, but rather passes through the point.

The first theorem refers to the multiplicity of the pseudo 0-sector at the point:

Theorem 1 *Let P and $C(t)$ be a point and a rational parametric curve in the plane, respectively. Let also k be the number of points of $C(t)$ at which the line orthogonal to $C(t)$ passes through P. Then the pseudo 0-sector of P and $C(t)$ passes k times through P.*

Note that the theorem refers to a number of points of $C(t)$ and not to a number of lines. This is because of multiplicity of lines: a line can be perpendicular to $C(t)$ at more than one point. Refer again to Figure 2. In Figure 2(a), the pseudo 0-sector of the point $P_1 = (0, 1)$ and the line $C_2(t) = (0, t)$ passes once through P_1. Indeed, only the line $y = 0$, the perpendicular to C_2 at $t = 0$, passes through P_1. In Figure 2(b), the pseudo 0-sector of a point P_2 and a curve passes three times through the point. This is because three different perpendicular lines to the curve pass through the point P_2. Finally, in Figure 2(c), the pseudo 0-sector of a point P_3 and a circle passes twice through the point. This is a multiplicity case: A line, which is perpendicular to the curve at two points, passes through the point P_3.

The second theorem refers to the slope of the pseudo 0-sector when it passes through the point:

Theorem 2 *Let P and $C(t)$ be a point and a rational parametric curve in the plane, respectively. Assume that P is the pseudo 0-sector point that corresponds to $C(t_0)$. Then the slope of the pseudo 0-sector at P and the slope of C at $C(t_0)$ are identical.*

6 Conclusion

In this paper we have extended the notion of a bisector of two varieties to the α-sector (for $0 \leq \alpha \leq 1$) of two varieties. We have also presented the pseudo α-sector, which approximates the α-sector by replacing the distance constraint with a linear version of it.

The interested reader may find more details in [EBK] and in the full version of this paper.

References

[Elb] G. Elber and M.-S. Kim, The bisector surface of freeform rational space curves, *ACM Trans. on Graphics*, 17 (1), 32–49, January 1998.

[EBK] G. Elber, G. Barequet, and M.-S. Kim, Bisectors and α-sectors of rational varieties, *Dagstuhl Seminar on Solid Modeling*, Dagstuhl, Germany, May 1999, submitted to *IEEE Computing.*