Optimal Tetrahedralizations Of Some Convex Polyhedra [1]

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Abstract: In this paper, we investigate the problems of finding the 'optimal' tetrahedralization of a convex polyhedron in terms of minimum number of tetrahedra. We identify a class of convex polyhedra, called *k-vertex emission tetrahedralizable* polyhedra and prove that if a polyhedron contains an optimal solution of this type (denoted by *k-opt*), then it can be found in polynomial time for fixed k.

1 Introduction

Given a convex polyhedron P, what is the optimal tetrahedralization in terms of minimum number of tetrahedra? This is a fundamental open problem in computational geometry [GOR97]. While the complexity status of this problem in general case is not known, researchers look for special classes of polyhedra whose optimal solution can be found in polynomial time. The only known non-trivial class is so-called *stacked* polyhedra [De99]. In this paper, we shall prove that the class of $k-opt$ polyhedra can be optimally tetrahedralized in polynomial time.

Let P be a convex polyhedron with vertex set V and edge set B, where $\mid V \mid = n$. Let G_B denote the planar graph with vertex set V and edge set B. Let T be any tetrahedralization of P and E be its edge set. $E_i = E - B$ is the internal edges of T. G_T is the graph with vertex set V and edge set E.

Definition: A tetrahedralization of P is *k-vertex emission* for $k < n$, denoted by T_k, if G_B can be partitioned into k simple cycles such that for each region bounded by a cycle, there exists a vertex v_i (called the *tip*) which is *adjacent* to every vertex of the region in G_{T_k}. Two regions, say $R(v_s)$ and $R(v_t)$ with tips v_s and v_t, are *neighborhood* if they share a piecewise edges on their cycles and if edge $\overline{v_s v_t}$ belongs to E_i (the edges internal to P). These tetrahedra of T_k, whose edges connect only the vertices in $R(v_s)$ (resp. in $R(v_t)$), are called *skin block* w.r.t. v_s (resp. v_t), denoted by SB_{st}. The remaining part of T after the removal of these skin blocks is called the *core* of T_k, denoted by C_{T_k}.

Figure 1 shows a 2-vertex emission tetrahedralization. Vertices v_1 and v_9 are two tip vertices, the shared edges of $R(v_1)$ and $R(v_9)$ are $\overline{v_2 v_6}$, $\overline{v_6 v_7}$, $\overline{v_7 v_5}$, and $\overline{v_5 v_2}$. The skin block $SB_{1,9}$ consists of $\triangle v_1 v_2 v_3 v_6$, $\triangle v_1 v_4 v_6 v_7$, and $\triangle v_6 v_7 v_8 v_9$, and the core consists of $\triangle v_1 v_9 v_2 v_6$, $\triangle v_1 v_9 v_6 v_7$, $\triangle v_1 v_9 v_7 v_5$, and $\triangle v_1 v_9 v_5 v_2$.

Definition: A class of convex polyhedra is *k-vertex emission optimal tetrahedralizable*, simply *k-opt*, if their optimal tetrahedralizations contain a k-vertex emission tetrahedralization.

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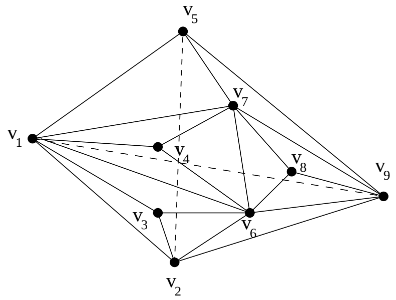

Figure 1: For the definitions

2 The optimality of k-opt polyhedra

It is well-known [De99] that the following equation describes the relationship between the number of tetrahedra t and the number of interior edges e_i in a tetrahedralization of P, $t = e_i + n - 3$.

Lemma 1 *The tip vertex of an optimal solution in any 1-opt polyhedron must have the largest degree in G_B.*

Proof: Let v be the tip vertex in a 1-vertex emission tetrahedralization of P, and let $d(v)$ be the degree of v at G_B. Then, the number of internal edges e_i is $n - d(v) - 1$ because in order to complete the tetrahedralization of P, all vertices except the adjacent vertices of v need connecting interior edges to v. Then, the lemma follows the fact that to minimalize t is to minimalize e_i and this in turn is to maximalize $d(v)$. $\square$

Lemma 2 *If P is a 2-opt polyhedron, then the optimal solution can be determined by finding*

$$max_{v_s, v_t \epsilon V}(d(v_s) + d(v_t) - \mid b(v_s) \cap b(v_t) \mid)$$

Proof: Let the two tip vertices be v_s and v_t in a 2-vertex emission tetrahedralizable polyhedron. Then, the number of interior edges in any $T_{s,t}$ is that $e_i = n - (d(v_s) + d(v_t)) - 1 + \mid b(v_s) \cap b(v_t) \mid$, where $d(v_s)$ and $d(v_t)$ are the degrees of the tip vertices v_s and v_t in G_B and $b(v_s) \cap b(v_t)$ is the edges on the cycles shared by $R(v_s)$ and $R(v_t)$. This is because each vertex in $b(v_s) \cap b(v_t)$ determines two interior edges connecting to either v_s or v_t and the terms: $n - (d(v_s) + d(v_t)) - 1$ only count one interior edge for each vertex in $b(v_s) \cap b(v_t)$. Therefore, the number of tetrahedra in $T_{s,t}$ is that

$t = (n - (d(v_s) + d(v_t)) - 1 + \mid b(v_s) \cap b(v_t) \mid) + n - 3 = 2n - (d(v_s) + d(v_t) - \mid b(v_s) \cap b(v_t) \mid) - 4.$ $\square$

Lemma 3 *If P is a k-opt polyhedron for $k \geq 4$, then the optimal solution can be determined by finding*

$$max_{V_k \subset V}(\sum_{v_s \epsilon V_k} d(v_s) - \sum_{i \neq j, v_i, v_j \epsilon V_k} d(v_i) \cap d(v_j) + N(c)).$$

Proof: If P is a k-vertex emission tetrahedralizable polyhedron, then the number of the interior edges in such a tetrahedralization is

$$e_i = n - (\sum_{v_s \epsilon V_k} d(v_s) - \sum_{i \neq j, v_i, v_j \epsilon V_k} d(v_i) \cap d(v_j)) - N(c)$$

where $N(c)$ is the number of edges internal to the core in T_k. When $k = 1$, the core is regarded as the tip vertex v, then $N(c) = 0$; when $k = 2$, the core is a two 'facing pyramids' with more than three shared edges, then $N(c) = 1$; when $k = 3$, the internal edges of the core may form a triangle (three such edges connecting each other) or only two edges, thus $N(c) \leq 3$; when $k = 4$, the internal edges of the core may form a tetrahedron or a set of its lower dimensional faces (triangle face or edge), thus $N(c) \leq 6$; in general, the internal edges of the core may form a polyhedron with k vertices or a set of its lower dimensional faces, thus $N(c) \leq \mid E_k \mid$, where E_k is the number of edges in this polyhedron with k vertices. In any case, for given P and given k vertices, $N(c)$ is fixed. Thus, to minimalize t is to minimalize e_i, and this in turn is to maximize the stated term. □

3 The algorithmic issue

In this section, we shall show that If the number k is a constant in a *k-opt* polyhedron, then the optimal tetrahedralization of P can be found in polynomial time. Let us first consider when $k = 1, 2, 3$.

Lemma 4 *The optimal tetrahedralization of a 1-opt polyhedron P can be found in $O(n)$ time.*

Lemma 5 *The optimal tetrahedralization of a 2-opt polyhedron P can be found in $O(n^5)$ time.*

Proof: Let v_i and v_j to be the tips of a 2-opt. We shall find the shortest cycle partitioning v_i and v_j on the surface of P, which will determine the optimal solution. Note that the desired cycle must cross the shortest path connecting v_i and v_j, denoted by $SP(v_i, v_j)$. Now, we split $SP(v_i, v_j)$ into two, i.e., duplicate each interior vertex and each edge of $SP(v_i, v_j)$, denote these two paths connecting v_i and v_j as $SP'(v_i, v_j)$. We form a new graph from G_B by replacing $SP(v_i, v_j)$ by $SP'(v_i, v_j)$ and then removing v_i, v_j, and all their incident edges. That is, $G' = G_B - SP'(v_i, v_j) - \{v_i, v_j\} - E_{i,j}$, where $E_{i,j}$ are the edges incident to v_i or to v_j. Then, we apply an all-pairs shortest path algorithm, say Floyd-Warshall algorithm ($\Theta(n^3)$) to G'. The shortest path connecting a pair of split vertices in $SP'(v_i, v_j)$ will be the shortest cycle partitioning v_i and v_j. Then, we can tetrahedralize P by performing 1-vertex emission from v_i to all vertices on the same side of this cycle and 1-vertex emission from v_j to all vertices on the other side, and connect v_i and v_j. By Lemma 2, we obtain an optimal tetrahedralization of P in $O(n^3)$ time. If the emitting vertices v_i and v_j are not known, then we have to test each pair of vertices in P. There are n-choose-two ways, that is $\frac{n(n-1)}{2}$ pairs. This yields an $O(n^5)$ time algorithm. $\Box$

In a 3-opt polyhedron, the partitions for these three tips, say v_i, v_j, and v_k can have two cases. One is that a pair of regions, say $R(v_i)$ and $R(v_j)$, do not share any boundary piece, i.e., the shared boundary of $R(v_i)$ and $R(v_k)$ and the shared boundary of $R(v_j)$ and $R(v_k)$ do not intersect. The other is every pair among $R(v_i)$, $R(v_j)$ and $R(v_k)$ shares some pieces of their boundaries. In the former case, it contains two 2-opt cases. In the latter case, there must exist two vertices, there exist two vertices, say v_s and v_t, each of them is shared by $R(v_i)$, $R(v_j)$ and $R(v_k)$. These shared vertices are called *meeting vertices*, simply m-vertex. Moreover, the shortest path connecting v_s and v_t will cross the shortest path connecting a pair of v_i, v_j, and v_k in G_B. Thus, there exist three such pairs of paths connecting v_s and v_t. They will optimally partition $R(v_i)$, $R(v_j)$, and $R(v_k)$ in terms of minimum number of shared boundary edges and hence by Lemma 3 they determine an optimal tetrahedralization of P.

Lemma 6 *The optimal solution of a 3-opt polyhedron can be found in $O(n^7)$ time.*

Proof: There are n-choose-three ways for the tip vertices, i.e., $\frac{n(n-1)(n-2)}{3}$ combinations. For each combination, we have n-choose-two ways for the two m-vertices, i.e., $\frac{n(n-1)}{2}$ combinations; and two combinations if no m-vertex. The total combinations will be bounded by $O(n^5)$. For the fixed tip-vertices and m-vertices, we apply the same method as in 2-opt case. That is to split the shortest paths among the tip-vertices and removing the tip-vertices as well as their incident edges and applying the all-pairs shortest paths algorithm to the resulting graph. This process takes $O(n^3)$ time. Thus, the total time is $O(n^7)$. $\Box$

In general case, for $k \geq 4$ there exist three components in the optimalization; namely the k tip-vertices, the m-vertices, and the core with k vertices.

Lemma 7 *The optimal solution of a k-opt polyhedron for $k \geq 4$ can be found in $(n^{k+3}(k^{2k^2}))$ time.*

Proof: There are n-choose-k ways for the tip vertices, i.e., $O(n^k)$ combinations. For each combination, we have $\frac{k(k-1)}{2}$ neighborhood relationships among the k tip-vertices. Each of such neighboring relationship can be either on or off. Thus, we have $2^{\frac{k(k-1)}{2}}$ such combinations, where each combination determines a set of m-vertices. The total combinations will be $O(n^k(2^{\frac{k(k-1)}{2}}))$, i.e. $O(n^k(2^{k^2}))$. For the fixed tip-vertices and m-vertices, we apply the same method as in 2-opt case. That is to split the shortest paths among the tip-vertices and removing the tip-vertices as well as their incident edges and applying the all-pairs shortest paths algorithm to the resulting graph. This process takes $O(n^3)$ time. Thus, the total time is $O(n^{k+3}(2^{k^2}))$. $\square$

Theorem 1 *The optimal solution of a k-opt polyhedron for fixed k can be found in polynomial time.*

4 References

[AE87] Avis D. and ElGindy H., *Triangulating point sets in space*, Discrete Comput. Geom. 2, (1987), pp. 99-111.

[BE92] Bern M. and Eppstein D., *Mesh generation and optimal triangulation* in Computing in Euclidean Geometry (Du D. and Hwang F. eds.), World Scientific Publishing Co. (1992).

[BFS90] Billera L., Filliman P., and Sturmfels B., *Constructions and complexity of secondary polytopes*, Adv. in Math., 83, 1990, pp. 155-179.

[C82] Cottle R., *Minimal triangulation of the 4-cube*, Discrete Methematics, 40, 1982, pp. 25-29.

[De99] De Loera J., *Computing minimal and maximal triangulations of convex polytopes*, Macrch 8, 1999. Manuscript. Institut fur Theoretische Informatik, ETH-Zentrum, Zurich.

[DLHSS96] De Loera J, Hosten S., Santos F., *The polytope of all triangulations of a point configuration*, Documenta Mathematica, 1, 1996, pp. 103-119.

[EPW90] Edelsbrunner H., Preparata F., and West D., *Tetrahedralizing point sets in three dimensions*, Journal of Symbolic Computation, 10, 1990, pp. 335-347.

[GOR97] Goodman J. and O'Rourke J, *Handbook of Discrete and Computational Geometry*, CRC Press, New York, 1997.