Some Separability Problems in the Plane *

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Abstract

We study some problems on the separability of two disjoint point sets in the plane by means of several strips, wedges and sectors. We also give an efficient algorithm for minimizing the number of misclassified points, when only one strip is used, and a lower bound for the fixed–slopes double wedge problem.

1 Introduction

Let B and R be two finite disjoint point sets in the plane; we refer to B as the set of *blue* points and R as the set of *red* points. We denote $n = |B|$, $m = |R|$ and $N = \max\{n, m\}$.

In image analysis, statistics, and in other fields in which discrimination-classification is required, it is useful to have some geometric separator: a finite set S of curves in the plane is a *separator* for the sets B and R if every connected component in $\mathbb{R}^2 - S$ contains objects only from B or only from R. We also say that each connected component is *monochromatic*. Separability can also be defined for sets of objects other than points and for objects in higher dimensions. It can also be extended to cases in which strict separability is not possible, as suggested by Houle [10]. Decision and optimization problems in this area have attracted much attention.

The most natural separability in the plane is by means of a line; when that is possible we say that B and R are *line separable*. It is well known that the decision problem of linear separability for sets of points, segments, polygons or circles can be solved in $O(N)$ time [10, 13], and that the locus of points p such that there is a separating line through p can be constructed in $O(N \log N)$ time.

Bhathacharya, Boissonnat et al., O'Rourke et al. and Fish have studied the case in which S is a circle (the *circular separability* problem), paying also attention to finding the smallest and largest such separators [2, 3, 14, 8]. Edelsbrunner and Preparata [6] considered the problem of convex polygonal separators, giving algorithms both for the decision problem and the problem of minimizing the number of edges in the separator.

A *wedge* is the union of two rays with common origin (the *apex*). A *strip* is the union of two parallel lines; their common slope is *the slope of the strip*. In [11] the authors give algorithms for deciding wedge separability and strip separability, as well as for constructing the locus of feasible apices and the interval of feasible slopes. Wedges with maximum/minimum angle, and the narrowest/widest strip are also found.

A *double wedge* is the union of two crossing lines. A *constant-turn polygonal chain* is a (simple) polygonal chain (the first and last edges are rays) in which all angles have the same absolute value but alternate in turn sense. Efficient algorithms for deciding the existence of double-wedge and constant-turn polygonal chain separators are given in [12]; they also give an algorithm for maximizing the angular constant turn, thereby, in a sense, determining the "straightest" possible chain separator (closest to being a line separator).

Other related separation problems include the minimum-cardinality *shattering* problem (find an arrangement of the fewest lines whose arrangement decomposes the plane into monochromatic cells [9]) and the *minimum-link red-blue separation* problem (find a polygonal separator with fewest edges that separate B from R [7]). Both problems are known to be NP-complete.

In this paper we describe efficient algorithms for (1) deciding the separability with two wedges or with two strips; (2) finding a minimum-cardinality separator set of parallel lines; (3) computing a minimum-cardinality set of separator rays having a common apex; and, (4) minimizing the number of misclassified points when only one strip is used. Finally, we prove an $\Omega(N \log N)$ lower bound for the fixed-slopes double-wedge separability problem.

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2 Strip Separability

In this section we extend the results in [11] to cases in which more than one strip, or more than two parallel lines, are used.

2.1 Several strips

The first natural extension to the separation with one strip is to use three parallel lines. The separability with one strip can be decided in $O(N \log N)$ time, and the intervals of feasible slopes can be constructed within the same running time. We have shown that the same bound applies when we use at most three (parallel) lines:

Theorem 1. *Let B and R be two disjoint point sets in the plane. To decide whether they are separable by at most three parallel lines and to compute the corresponding intervals for the slopes of the separating lines can be done in $O(N \log N)$ time.*

We say that two strips separate B and R if all of the red points are inside the strips, and all the blue points are outside, or conversely (Figure 1). If such a separating pair of strips exists, the *two widest separating strips* can be defined in various ways, e.g., maximizing the minimum or the maximum of the widths, or the sum of the widths. The two *narrowest* strips are defined similarly.

Since one of the strips can be perturbed until one of the bounding lines contains a point from B and a point from R, it is possible to check tentatively the configurations combinatorially different for one of the strips, and to use the one strip separability algorithm for the remaining points. In this way we obtain:

Theorem 2. *The separability by two strips of the sets B and R can be determined in $O(N^3 \log N)$ time. In the same time the two narrowest and the two widest separating strips can be found (either sense above).*

The running time decreases when we constrain the problem by using two fixed strip slopes (e.g., horizontal and vertical, as in Figure 1c), or a fixed slope combined with a free slope:

Theorem 3. *If the slope of one of the strips is given, the separability by two strips of the sets B and R can be determined in $O(N^2 \log N)$ time; in the same time the two narrowest and the two widest separating strips can be found. If the two slopes are given, the running time can be reduced to $O(N)$.*

2.2 Minimum number of strips

We consider here the problem of finding the minimum number of parallel strips (or the minimum number of parallel lines) separating B and R, and the intervals of slopes corresponding to the solution.

A related problem was studied in [1]: Given a 2-coloring of the vertices of a regular N-gon, how many parallel lines are needed to separate the vertices into monochromatic subsets? It is shown that $\lfloor N/2 \rfloor$ is a tight upper bound, and an $O(N \log N)$ algorithm for finding the direction that gives the minimum number of lines is given.

For the general case we consider here we have:

Theorem 4. *The minimum number of parallel strips (or parallel lines) separating sets B and R, and the corresponding slope intervals, can be determined in $O(N^2 \log N)$ time.*

2.3 Misclassified points with one strip

If there is no strict separability with a given class of separator, it is natural to weaken the separation constraint, allowing the existence of some misclassified points, while trying to minimize their number. This concept is related to the *bichromatic discrepancy*, as described in [5].

Everett et al. [4] give an $O(Ns \log s + N \log N)$ algorithm for finding a line that weakly separates the sets B and R in such a way that the sum, s, of the number of red points above the line and the number of blue points below the line is minimized.

Here, we consider a similar problem for weak separability with one strip, and obtain the following result:

Theorem 5. *For disjoint point sets, B and R, a strip that contains all the points from one of the sets and the minimum number of points from the other set, can be constructed in $O(N \log N)$ time.*

3 Wedge Separability

In this section we extend the results in [11] to cases in which more than one wedge is used.

3.1 Several wedges

A wedge partitions the plane into two regions, one of them convex (the *interior* of the wedge). We say that two wedges separate B and R if all of the red points are inside the wedges, and all of the blue points are outside, or conversely (Figure 2).

We consider two cases, depending on whether the wedges intersect or not, and obtain the following:

Theorem 6. *The separability by two wedges of the sets B and R can be determined in $O(N^5 \log N)$ time. If the wedges must be disjoint, the separability can be decided in $O(N^3 \log N)$ time.*

3.2 Minimum number of sectors

The high complexities in the preceding theorem come from the degrees of freedom of the wedges. A constrained version (the *sector separability* problem) occurs when all of the wedges have the same apex.

Observe that this kind of separation is always possible; the interesting issue is to minimize the number of rays used. A more constrained version is to force the common apex to lie in a given region (Figure 3). Our results include the following:

Theorem 7. *Let B and R be two disjoint point sets in the plane. A minimum-cardinality set of rays (having common apex) separating B and R can be computed in $O(N^4)$ time.*

Theorem 8. *Let B and R be two disjoint point sets in the plane, and let l be a line. A minimum-cardinality set of rays separating B and R, with common apex on l, can be computed in $O(N^2 \log N)$ time. The locus of possible placements in l can be obtained within the same time bound.*

Theorem 9. *Let B and R be two disjoint point sets in the plane, and let l be a line. To decide whether B and R are separable or not by means of at most three rays from a common apex on l can be done in $O(N \log N)$ time. The locus of possible placements on l can be obtained within the same time bound.*

3.3 Double wedge with fixed slopes

A *double wedge* is the union of two crossing lines. An $O(N \log N)$ algorithm for checking double-wedge separability is given in [12].

When the slopes of the lines are given (say horizontal and vertical, as in Figure 4), it is reasonable to expect that a faster algorithm may be possible; however, we prove (using a reduction from MAXIMUM GAP) the following:

Theorem 10. *Let B and R be two disjoint point sets in the plane. Deciding whether these sets are separable with a double wedge with given slopes requires $\Omega(N \log N)$ time.*

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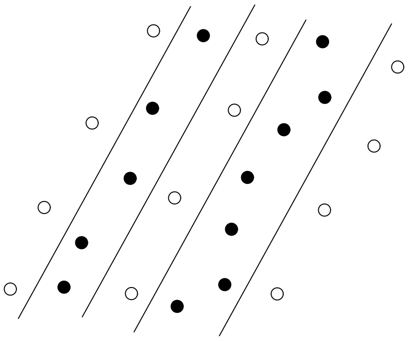
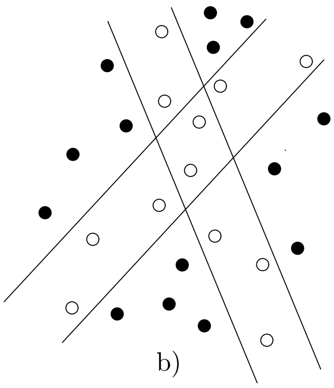


a)



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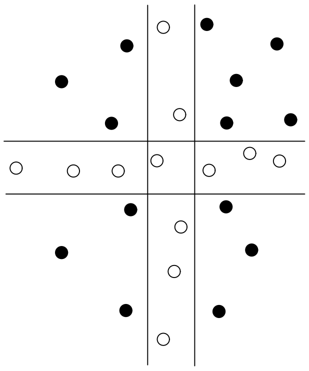
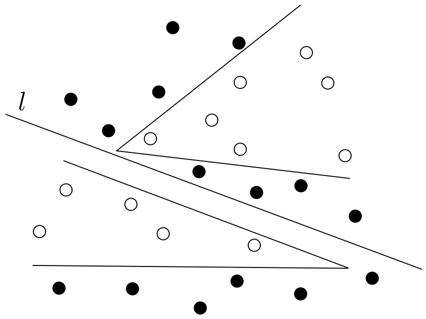


c)

Figure 1: a) Two parallel strips, b) Two strips, c) vertical-horizontal strips.



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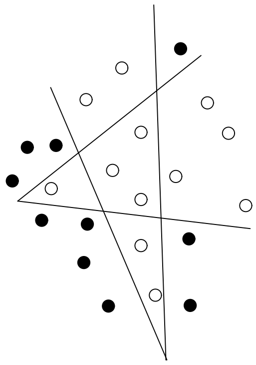

b)

Figure 2: a) Two disjoint wedges, b) Two wedges.

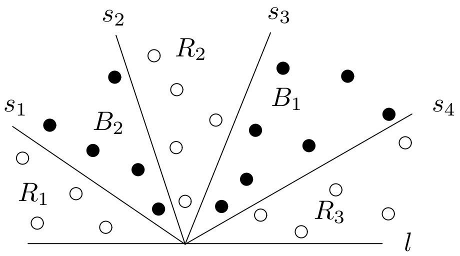

Figure 3: Sector separability from a line.

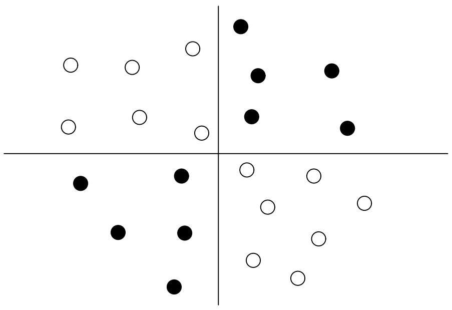

Figure 4: Vertical-horizontal double wedge.