On the approximation of polygons by subpolygons

Peter Brass*

Institute for Computer Science, FU Berlin,
Takustrasse 9, D-14195 Berlin, Germany
brass@inf.fu-berlin.de

(Extended Abstract)

1. Introduction

The approximation of polygons or convex sets by polygons with few vertices is an old and much-studied subject of discrete geometry. And at least without the convexity assumption it is also a problem of big practical importance: scanning or database operations frequently result in polygons with a very big number of vertices, which have to be 'simplified' for further use, where simplification means the construction of some polygon approximating the original polygon and having a small number of vertices. Thus it is a subject treated under different aspects in discrete geometry (see e.g. survey [5]), computational geometry ([1][2][6][7][9]) and image processing ([3][10]).

The setting of such an approximation problem requires that we specify

- a class of polygons to be approximated, like simple polygons or convex polygons,
- a class of admissible approximating polygons, like arbitrary k-gons, inscribed or circumscribed k-gons, sub-k-gons (involving a subset of the vertices), and
- a distance measure, like the symmetric-difference metric, the perimeter deviation, the Hausdorff- or Eggleston-metric, the L^p-metric of the support functions or the Frechét-metric of the parametrizations.

In any such setting we can ask for the best approximation by a polygon of given vertex-number (min-ε problem), or the minimum vertex-number of a polygon with given approximation quality (min-# problem); we can ask for algorithms to find these polygons and for a-priori bounds on the approximation error of these polygons. In the following we will be especially interested in the approximation of convex polygons by subpolygons (whose vertices are a subset of the vertex-set) of given vertex-number k with respect to the Hausdorff-metric, and in error bounds for that setting.

2. Extremal Configurations and Error Bounds

It is the aim of this note to study the configurations that are worst approximable in this setting; for if we know the worst case, this immediately implies an error bound for all other cases. Without any further restriction, however, this is not a meaningful problem, since we can scale the polygon to be approximated and thus achieve arbitrary big approximation errors. So we have to fix a class $\mathcal{P}$ of admissible polygons, e.g. polygons of area at most one, or of diameter at most one, or polygon restricted by some container. But we believe that the structure of the extremal set is independent of our particular choice of $\mathcal{P}$:

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Conjecture: In any class $\mathcal{P}$ of admissible convex polygons that is closed under selection of subpolygons the polygon $P \in \mathcal{P}$ that is worst Hausdorff approximable by a sub-k-gon is a $k+1$-gon.

One should compare this with the corresponding result for the symmetric-difference metric (where it is natural to take as admissible sets those of at most a given area):

Theorem [4]: If X is a set of points in the plane that is worst approximable by a sub-k-gon in the class $\mathcal{X} = \{X \mid \text{area}(\text{conv}(X)) \leq 1\}$ with respect to the symmetric-difference metric, then $\text{conv}(X)$ is an ellipse.
For any set X and any $k \geq 3$ there is a $Y \subset X$ with $|Y| \leq k$ such that

$$\frac{\text{area}(\text{conv}(Y))}{\text{area}(\text{conv}(X))} \geq \frac{k}{2\pi} \sin \frac{2\pi}{k}.$$

This result was obtained for $k = 3$ by Blaschke in 1923 and for arbitrary k by Sás in 1939, and similar extremal properties of the ellipse for the symmetric-difference metric are known in a number of other settings. Indeed the symmetric-difference metric has been much studied, since these bounds have some applications in packing- and covering theory (e.g. [4]), and since the symmetric-difference metric behaves in some ways much nicer than the Hausdorff metric, e.g. the sequence of approximation errors by k-gons, $k = 3, 4, \ldots$, is convex (Dowker-type theorems, not true for Hausdorff metric).
The Hausdorff metric seems to be algorithmically much more interesting ([1][2][6][7][9]), but although there are asymptotic bounds on the approximation error ([4][8]), there is no comparable result on the extreme cases for fixed k.

It is simple to prove our conjecture in some special cases:

Theorem The conjecture is true if $k = 3$ and $\mathcal{P}$ is the set of polygons contained in a given disk, or the set of polygons with at most a given diameter.

Unfortunately in general we can prove only a statement weaker than our conjecture:

Theorem In any class $\mathcal{P}$ of admissible convex polygons that is closed under selection of subpolygons the polygon $P \in \mathcal{P}$ that is worst Hausdorff-approximable by a sub-k-gon is an l-gon with $l \equiv 1 \bmod k$.

This holds even locally: if a set X cannot be approximated by a sub-k-gon Y with a given Hausdorff-error, then X has a sub-l-gon $\hat{X}$ that also cannot be approximated, such that $l \equiv 1 \bmod k$ and $\hat{X}$ has a very special structure.

Theorem Let $\varepsilon > 0$ and $k \geq 3$.
If X is a set such that $d_{\mathrm{H}}(\text{conv}(X), \text{conv}(Y)) > \varepsilon$ holds for all $Y \subset X$ with $|Y| \leq k$, then there is a subset $\hat{X} \subset X$ with $|\hat{X}| \equiv 1 \bmod k$ such that $d_{\mathrm{H}}(\text{conv}(\hat{X}), \text{conv}(\hat{Y})) > \varepsilon$ holds for all $\hat{Y} \subset \hat{X}$ with $|\hat{Y}| \leq k$.

If $|\hat{X}| = ak+1$ for some $a \geq 2$, then it is a polygon $x_1, \ldots, x_{ak+1}$ of the following structure: for each i and j with $i < j < i+a$ we have $d(x_j, \overline{x_i x_{i+a}}) \leq \varepsilon$, and there is a $b \leq a$ such that $d(x_{i+b}, \overline{x_i x_{i+a+1}}) > \varepsilon$ for all i, and $d(x_j, \overline{x_i x_{i+a+1}}) \leq \varepsilon$ for $j \in \{i+1, \ldots, i+a\} \setminus \{i+b\}$. We believe such polygons can exist only if $2b = a+1$, in which case the required sub-$k+1$-gon can be selected.

3. References

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