Convex hull in non-planar surfaces

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Being a basic and natural concept, the convex hull has many applications as well as a rich mathematical theory behind it. The computation of the convex hull of a set has been studied extensively and has applications, for example, in pattern recognition [1], image processing [8] and stock cutting and allocation [3, 9, 4]. Motion planning is another applied subject for wich convex hull provides an appropiate tool, for example, in collision avoidance [7], and a subfield of motion planning of considerable practical interest is planning the motion of an articulated robot arm. However, in most of the cases, the trajectory of the arm is not in the plane, but it is in a non-planar surface. Then, it is interesting to consider the convex hull on some surfaces, as the cylinder and the torus.

Several extensions of convexity to non-planar surfaces (or to non-euclidean spaces) has been considered in the literature. Most of them are based on metrical concepts, and more concretely, in the family of geodesics of the surface. In this talk, we deal with those extensions in the case of two of the simplest surfaces, the cylinder and the torus.

In order to describe the family of geodesics, as usual, we identify the cylinder with the quotient space obtained from the plane by identifying those points with the same ordinate such that their abscisae differ in an integer number. And the flat torus that is obtained from the plane by identifying those points such that both their abscisae and ordinates differ in an integer number. With the metric obtained from this definition, the geodesics joining two points in both surfaces can be identified with the segments in the plane joining a fixed representative of one of the points and all of the representatives of the other point. We say that generatrices $\{(x, y) : x = x_0\}$ and $\{(x, y) : x = x_0 + 1/2\}$ (parallels$\{(x, y) : y = y_0\}$ and $\{(x, y) : y = y_0 + 1/2\}$) are *opposite generatrices (parallels)*.

Using this representation, we can define the *strip* of a set of points $P = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ in the cylinder with $y_1 \leq y_2 \leq \ldots \leq y_n$ as the open strip O delimited by the maximal circles in the extreme points of P with respect to the ordinates ($O = \{(x, y) : y_1 < y < y_n\}$). Equally, we define the *h-top* of P as the maximal circle $\{(x, y) : y = y_n\}$ if $y_{n-1} = y_n$ or the single point (x_n, y_n) otherwise (equivalently the *h-bottom*). And the *m-top* of P as the minimal arc containing all points of P with the ordinate y_n if that arc is shorter than a half of the circle or the whole maximal circle if that arc is greater than a half of the circle or the single point (x_n, y_n) otherwise (equivalently the *m-bottom*).

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Then, following [2], for a given surface S, we can say that $C \subseteq S$ is *hyperconvex* if given two points of C all geodesics in S joining those points are contained in C. And, as usual, given a set P of points in S, the *hyperconvex hull* of P is the smallest hyperconvex set containing P.

In general, this definition is too stringent as the following theorem shows

Theorem 1 *The hyperconvex hull of a set P of two or more points on S is the open strip delimited by the points P union the h-top and the h-bottom of P if S is the cylinder or the whole surface if S is the torus.*

In the theorem above, the convex hull of a set of points is too big, thus it is convenient to use another definition of convex. Then, as Hopf-Rinow's Theorem [5] proves that there exists always the shortest geodesic joining two points, we can define as in [6] that $C \subseteq S$ is *metrically convex* if given two points of C the minimum geodesic in S joining those points is contained in C. And, as usual, given a set P of points in S, the *metrically convex hull* of P is the smallest metrically convex set containing P.

It is possible to give the following characterizations of the metrically convex hull

Theorem 2 *The metrically convex hull of a set of N points P in the cylinder is*

1. *The convex hull of P in the plane if P is contained between two oppossite generatrices.*
2. *The open strip delimited by the points P union the m-top and the m-bottom of P otherwise.*

Moreover, this metrically convex hull can be computed in $O(N \log N)$ time in the first case and in linear time in the second case, and it can be decided in which one of the cases we are in linear time.

Theorem 3 *The metrically convex hull of a set of N points P in the torus is*

1. *The convex hull of P in the cylinder if P is contained between two oppossite generatrices or two opposite parallels.*
2. *The whole torus otherwise.*

Moreover, it can be decided in which one of the cases we are in linear time.

Observe that although the complexity in the worst case is the same as in the plane, in most of the situations the points will be in the second case of Theorems 2 and 3 , in fact

Proposition 1 *If N points are chosen uniformly and independently at random from a cylinder (torus), they will be in the second case of Theorem 2 (Theorem 3) with a probability $P = 1 - N/(2^{N-1})$.*

Thus, Proposition 1 says that in most of the cases it is possible to compute the metrically convex hull in linear time, but in those cases the convex hull so obtained is too big, in fact with the probability given in Proposition 1 the hyperconvex hull and the metrically convex hull will agree.

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