Geometric Tree Graphs

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Given a connected graph G, the tree graph $T(G)$ is defined as the graph having as vertices the spanning trees of G, and edges joining two trees T_1 and T_2 whenever $T_2 = T_1 - e + f$ for some edges e and f. Tree graphs enjoy a number of important properties, like being Hamiltonian [2, 6] and having maximum connectivity [8]. Additional results on tree graphs have been obtained recently [4].

Here we consider a geometric version of the problem. Given a set P of points in the plane, let $\mathcal{T}(P)$ be the set of non-crossing spanning trees of P (edges are straight line segments and do not cross). We define the *geometric tree graph* $T(P)$ as the graph having $\mathcal{T}(P)$ as vertices and the same adjacencies as in combinatorial tree graphs, that is, two non-crossing spanning trees T_1 and T_2 of P are adjacent if $T_2 = T_1 - e + f$. Geometric tree graphs have appeared previously in the work of Avis and Fukuda [1] as a tool for enumerating spanning trees. They show that $T(P)$ is connected for any point set P in general position and has diameter bounded by $2n - 4$ if $n = |P|$.

In this paper we concentrate on the combinatorial properties of the graphs $T(P)$ in the case where P is a point set in convex position. For any $n \geq 2$, let G_n be the geometric tree graph of a set of n points in convex position. As mentioned above, we know that G_n is a connected graph and has diameter at most $2n - 4$. It is also known [3, 9] that it has $\frac{1}{2n-1}\binom{3n-3}{n-1}$ vertices, and that every geometric tree graph of n points has at least this many vertices [5]. We finally remark that, using results from [4], one can obtain tight bounds for the chromatic number of G_n.

In this paper we have obtained a number of new results about the graphs G_n. They are summarized in the list below.

1. The diameter of G_n is at least $3n/2 - 5$.
2. The center of G_n (the set of vertices of minimum eccentricity) has $2n$ elements, the n stars and the n paths having all its edges on the convex hull.
3. The group of automorphisms of G_n is the group D_n of symmetries of a regular n-gon.

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4. The vertex-connectivity of G_n is equal to the mininum degree, which in turn is equal to $2n-4$.

5. The graph G_n is Hamiltonian.

The main tool in proving the last two results is a recursive construction of the graphs G_n, in which a tree T in G_n gives rise to $\binom{d+2}{2}$ different trees in G_{n+1}, where d is the degree of the n-th vertex in T. This kind of construction has proved useful in solving similar problems for graphs of triangulations instead of tree graphs [7].

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