Configurations between geometry and combinatorics

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1 Introduction

The word "configuration" is quite often used in mathematics, in particular in combinatorics, as a synonym for substructure. However, it is a well defined combinatorial structure like a graph or design. Moreover, it is one of the oldest combinatorial structures at all, defined already in a geometrical context in 1876. A modern definition is as follows.

Definition 1.1 *A configuration (v_r, b_k) is a finite incidence structure of v points and b lines such that*
(i) there are k points on each line and r lines through each point, and
(ii) there is at most one line through two given points.

Fur further information see [1]; there are only two other books which contain information on configurations [8] and [6], the latter of them was reprinted recently.

2 Configurations born in geometry

The word configuration in the above sense was used in Theodor Reye's book *Geometrie der Lage* for the first time. In the second edition of his book of 1876 [9] Reye mentioned configurations as mathematical objects in their own right as generalizations of the Desargues configuration 10_3. Theodor Reye was born in Ritzebüttel near Cuxhaven in 1838, was a professor of mathematics in Zürich, Aachen, and (after 1872) in Straßburg, and died in Würzburg in 1919.

A few years later a formal definition for configurations n_i was given in [10].

> Eine Configuration n_i in der Ebene besteht aus n Punkten und n Geraden in solcher Lage, dass jede der n Geraden i von den n Punkten enthält und durch jeden der n Punkte i von den n Geraden gehen.

This definition corresponds to the situation where $v = b = n$, i.e. the number of points is equal to the number of lines. In the plane the axiom that through two different points there is at most one line is automatically fulfilled. One year earlier already S. Kantor [7] had constructed all 10 configurations 10_3. These are combinatorial structures having the same parameters like the Desargues configuration but are pairwise nonisomorphic. Although the constructions of Kantor are done from a combinatorial point of view and the geometrical aspect is not touched at all in his paper, at the end of the journal 10 drawings of these 10 configurations in the plane are included. However, one of these drawings (see Fig. 1) is wrong (one line consists of two straight segments not having an angle of 180 degrees). A few years later it was proved by Schroeter for the first time that exactly one of these 10 configurations cannot be drawn in the plane. However, for Kantor it seemed to be not clear whether there is a difference between the combinatorial and the geometrical aspect of a configuration.

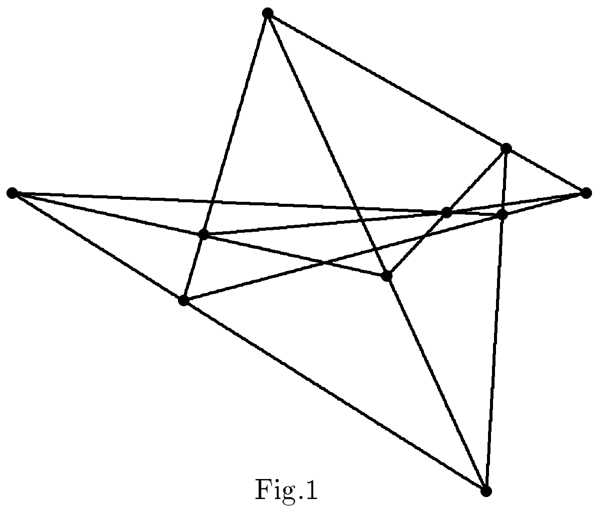

Fig.1

At the latest in Martinetti's paper of 1887 it is clear that configurations are only considered from a combinatorial point of view. Martinetti found a recursive construction method for configurations n_3 which does not care at all about geometrical properties. By using this method he constructed all configurations 11_3. For further details on the historical development see [2] and [3].

3 Configurations as a combinatorial structure

Independent from their geometrical background configurations (v_r, b_k) are defined as in Definition 1.1. There are two main questions for given parameters v, r, b, k. Does a configuration (v_r, b_k) exist? How many nonisomorphic of these configurations are there?

At least the following obvious necessary conditions must hold:

$$vr = bk, v \geq r(k-1)+1, b \geq k(r-1)+1.$$

However, they are not sufficient. For example, there is no configuration 22_5.

Concerning the problem of determining all nonisomorphic configurations with parameters v, r, b, k already in the last century all configurations n_3 with $n \leq 11$ were constructed. During the last ten years further results have been obtained. For example, the number of configurations 14_3 is 21399; the number of configurations $(12_4, 16_3)$ is 574.

Further particular properties of configurations have been discussed, mainly those of combinatorial character.

4 Realizations and drawings

¿From the point of view of pure combinatorics the problems of realizing and drawing configurations may be artificial. However, because of the origin and the early history of configurations they are quite natural (compare Section 2). Realizations and drawings have already been discussed in [5] and [4].

In order to find a drawing of a configuration in the plane it is necessary to assign coordinates to the points such that points are collinear in the plane if and only if they are on a common line of the configuration. Since the problem of checking the correctness of a drawing is not so easy (Figure 1), it seems to be better to compute the collinearity by means of algebra. On the other hand, also a drawing where the lines are nearly straight lines can serve as a good illustration of the configuration.

In fact, a correct drawing can be regarded as a projection of a 3-dimensional realization of the configuration over the field of the rationals or reals. This leads to an easy generalization in the following way.

Definition 4.1 *A realization of a configuration with point set P and line set L over a field K is a mapping from P to K^3 such that for all distinct $i, j, k \in P$*
$det(x_i, x_j, x_k) = 0$ *if and only if i, j, k are collinear.*

As an example the following 3-by-11 matrix realizes a certain configuration 11_3.

$$\begin{vmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 17 & 1 & 1 & 37 & 4 \\ 0 & 0 & 1 & 0 & 1 & -17 & 17 & 51 & 2 & 18 & -9 \\ 0 & 1 & 0 & 1 & 1 & 100 & 185 & 285 & 5 & 45 & 0 \end{vmatrix}$$

By computing all 3-by-3 subdeterminants over the rationals the configuration contains exactly the following 11 lines.

1,2,4 2,3,5 3,4,6 4,5,7 5,6,8 6,7,9 7,8,10 8,9,11 1,9,10 2,10,11 1,3,11.

In this way, the problem of realization of configurations now belongs to computational geometry, a quite modern part of mathematics. However, its origin is closely related to the origin of configurations and of modern geometry of the 19^{th} century.

References

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