The Path of a Triangulation
(Extended Abstract)

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1 Introduction

In this paper we introduce a new concept called *triangulation path* or *t-path* for short. Consider a planar point set S, a triangulation T of S and a line l properly intersecting T. There are two edges of the convex hull of S intersected by l. We define a *path* as a monotone "zig-zag" chain of edges connecting these two convex hull edges. Thereby zig-zag means that the vertices of the chain (from S) lie alternately to the left and right of l, and monotone means that the intersections with l are in a monotone order. It is easy to show that for any triangulation T of S there exists such a chain with respect to l. Moreover we may require some additional properties, i.e. the area bounded by the chain and l has to be free of points from S. In this case we will show that any triangulation of S has exactly one t-path for each line l.

We give two applications of t-paths. The first uses the t-path to enumerate or just count all possible triangulations of a given planar point set. In the second application we use a branch-and-bound scheme to compute optimal triangulations. Here the t-path proves useful in the "branching" part.

2 The Path

Let S be a planar point set and T a triangulation of S. A subset c of edges from T of size k is a *chain* with *length* k if there exists a labeling $e_1 \ldots e_k$ of the edges in c such that any subsequent pair e_i and e_{i+1}, $1 \leq i \leq k-1$, shares a common endpoint. The edges e_1 and e_k are called start and end edge, respectively.

Note that by the definition of a triangulation the edges of a chain do not properly intersect. On the other hand several edges may share the same vertex, i.e. a chain may contain circles. Clearly the length k of a chain is at most linear in the number of vertices in S. More precisely we get $k \leq |T| \leq 3n-6$ where n is the size of S.

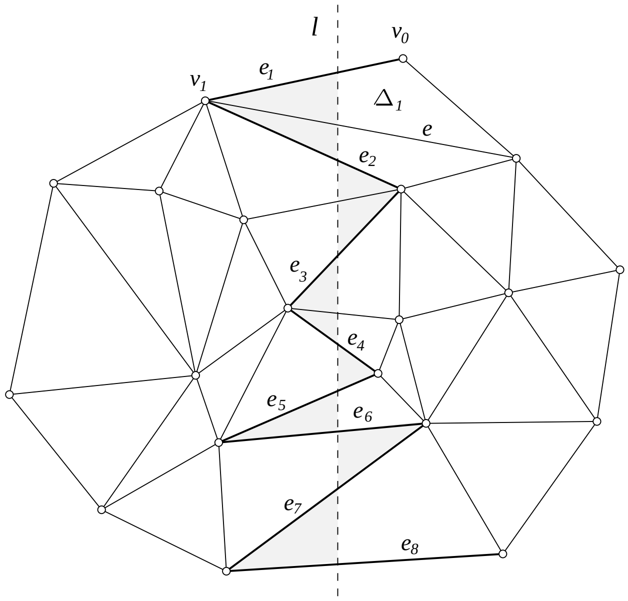

Figure 1: The path of a triangulation with respect to a line l. Shaded areas have to be free of points.

We are now ready to define a triangulation path (refer to Figure 1):

Definition 1 *Given a planar point set S, a triangulation T of S and a line l, $l \cap S = \emptyset$, properly intersecting the convex hull of S. The* triangulation-path (t-path) of T with respect to l, $p_l(T)$, *is defined as follows:*

1. *$p_l(T)$ is a chain of T where every edge of*

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$p_l(T)$ *intersects* l.

2. *Start and end-edges of* $p_l(T)$ *are the two edges of the convex hull of* S *intersected by* l.

3. *The (eventually multi-partitioned) area bounded by* $p_l(T)$ *and* l *must be free of points from* S *(refer to the shaded area in Fig. 1).*

Note that the length of a chain of a triangulation and thus the size of a t-path is at most linear in $|S|$. We first show that one can always construct such a t-path.

Theorem 1 *For a given planar point set* S*, a triangulation* T *of* S *and a line* l*,* $l \cap S = \emptyset$*, properly intersecting* T*, there exists always a t-path* $p_l(T)$*.*

The proof of Theorem 1 (which we omitted in this abstract) gives us a deterministic way to construct a t-path $p_l(T)$. The next theorem shows that this is in fact the only way to do so.

Theorem 2 *For a given planar point set* S*, a triangulation* T *of* S *and a line* l*,* $l \cap S = \emptyset$*, properly intersecting* T*, there exists exact one t-path* $p_l(T)$*.*

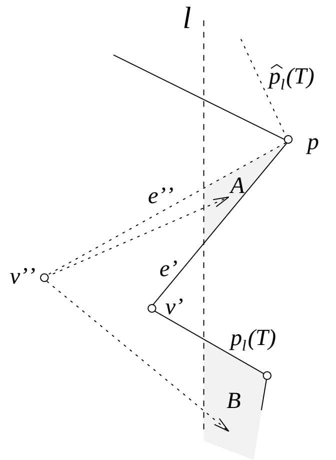

Figure 2: $p_l(T)$ and $\hat{p}_l(T)$ can not be both a t-path.

Proof omitted.

From the construction used in the proof of Theorem 1 and the uniqueness given in Theorem 2 we immediately get the following.

Observation 1 *The intersections of* $p_l(T)$ *with* l *are in monotone order, i.e. the distance of subsequent intersections to the first intersection is strictly increasing.*

Note also that from Theorem 2 it follows that the construction of the t-path does not depend on which side of the convex hull we start.

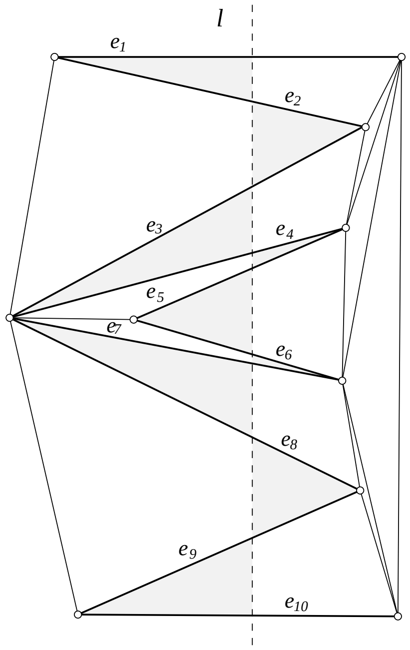

Figure 3: The path $p_l(T)$ may contain cycles.

That a t-path may indeed contain cycles is shown in Figure 3. Note that all other chains which connect the two intersected edges of the convex hull would violate Definition 1.

2.1 Computing the t-path

In this section we give an $O(\log n + m) = O(n)$ algorithm to compute the t-path for given S,T and l, where m is the number of edges of T intersected by l. The algorithms follows the lines of the proof of Theorem 1 and is triangle based. We assume that the triangulation T is stored in a way such that for any edge e of T we find its corresponding triangles of T in constant time and vice versa.

The conceptual sketch of the algorithm is summarized below.

1. Let $e = (v_l, v_r)$ be the first edge intersected by l, such that v_l is to the left of l and v_r to the right, respectively. Let v' be the third vertex of the triangle (in T) based on e.

2. If v' lies to the left of l THEN set $p = v_r$
ELSE set $p = v_l$

3. Output e

4. Set $p' = v'$ and $e = (p, p')$
5. WHILE $e \notin KH(S)$ DO
   Let v' be the third vertex of the triangle "below" e, based on e.
   IF v' and p lie on the same side of l
      THEN Output e
         Set $p = p'$ and $p' = v'$
         Set $e = (p, p')$
      ELSE Set $p' = v'$
         Set $e = (p, p')$

The correctness of the algorithm follows directly from Theorem 1. For the while loop we consider each edge of T intersecting l at most once and every line within the while loop can be done in constant time. (Note that to check whether an edge under consideration is a convex hull edge can be simply done by testing if there is only one triangle in T based on this edge. Thus by the above assumptions this can be done in constant time.) The initial search for the first edge of the convex hull intersected by l gives the logarithmic factor.

3 On different t-paths of a given point set

In this section we will give some applications of the t-path concept. To this end we first give some properties of different t-paths for a fixed point set. Not that in contrast to the previous section here we have only given the planar point set S and a line l intersecting the convex hull of S, but we do **not** consider a single fixed triangulation T of S.

Lemma 1 *Given a planar point set S, two triangulations T_1 and T_2 of S and a line l intersecting the convex hull of S. Then the two t-paths $p_l(T_1)$ and $p_l(T_2)$ are either identical or properly intersect each other.*

Proof omitted.

The merit of this lemma is that for given S and l a t-path can not lie entirely on one side of another t-path. Thus if we split (the area of the convex hull of) S with a t-path into distinct parts this guarantees that non of these parts can be a proper subset of a subpart of S obtained from the splitting with another t-path.

Let $p_l(S)$ be the number of different t-paths of S with respect to l and $t(S)$ the number of different triangulations of S. From Theorem 2 we immediately get

Lemma 2 $p_l(S) \leq t(S)$

3.1 Applications

In this section we will show how to use $p_l(T)$ to count the number of possible triangulations of a given planar point set S. From the last section it follows that we can do this in a Divide & Conquer like manner. First we separate the point set S into two halfs by a line l. Then we compute a possible t-path p_l for S and l and get thus two independent subproblems. We can now (recursively) compute the number of triangulations of these two subsets and multiply the results. This gives us the number of possible triangulations of S containing the t-path p_l. We now sum up these numbers over all possible t-paths of S and l and get thus the number of different triangulations of S. From Lemma 1 it follows that two different t-paths of S and l can never be part of the same triangulation of S and thus we can not count triangulations twice in this way. Since any triangulation of S contains at least one t-path for l we can also not miss a triangulation. This shows that we indeed count the exact number of triangulations of S.

From the fact that any t-path splits the set S into two subparts which we can treat independently we hope that this method is more efficient than straight forward enumeration of all triangulations.

Instead of just counting the number of triangulations this method may also be used to search for an optimal triangulation. Therefore it is necessary that the optimization criteria is "semi-local" in the sense that the "cost" of a fixed subpart of a triangulation is not effected by changing other parts of it. This is e.g. the case for the well known minimum weight triangulation (MWT), which minimizes the sum of the edge length. Thus we can optimize the triangulation containing a given t-path by optimizing the two subparts independently. Again we hope that the t-path provides a new approach to more efficient solutions of these problems.