# Seminar on Randomized Algorithms

# Game-Theoretical Applications Part III

By Ibraguim Kouliev

In the following discussion we explore techniques for finding **lower bounds** for running times of randomized game tree algorithms, and also address the general issues of **randomization** and **non-uniformity** in algorithms.

## Lower Bounds

First, we'll discuss a method for obtaining lowest possible bounds for randomized game tree evaluation algorithms.

A randomized game tree evaluation algorithm can be viewed as a probability distribution over deterministic game-tree algorithms applied to random game tree instances. Thus, we can use **Yao's Minimax principle** for our task.

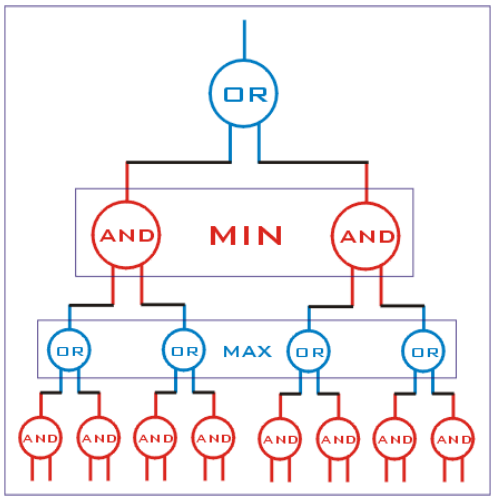
The **Yao's Minimax** principle is essentially based on the following observation:

*To find the lower bound for a randomized algorithm, it is sufficient to specify a probability distribution on its input data and then prove a lower bound on the expected running time of any deterministic algorithm operating on these data.*

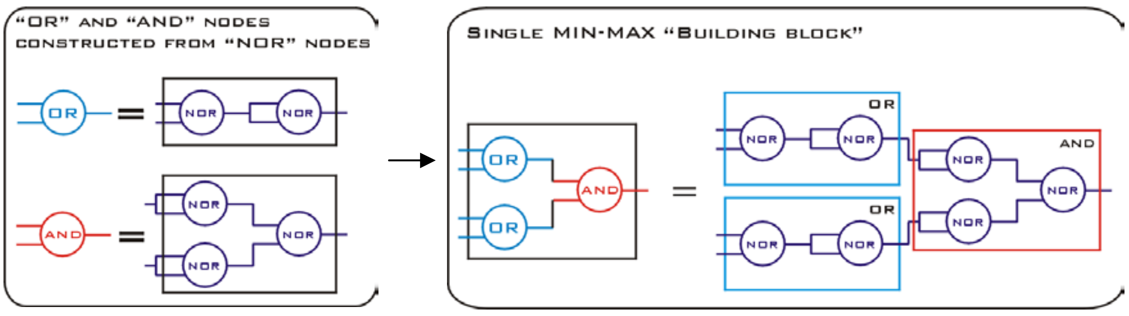
(One should note that this statement holds ONLY for Las Vegas algorithms!)

In our case, this means specifying a probability distribution on the values of the leaves of a game tree and then finding the lower bound for any deterministic game-tree evaluation algorithm applied to this tree.

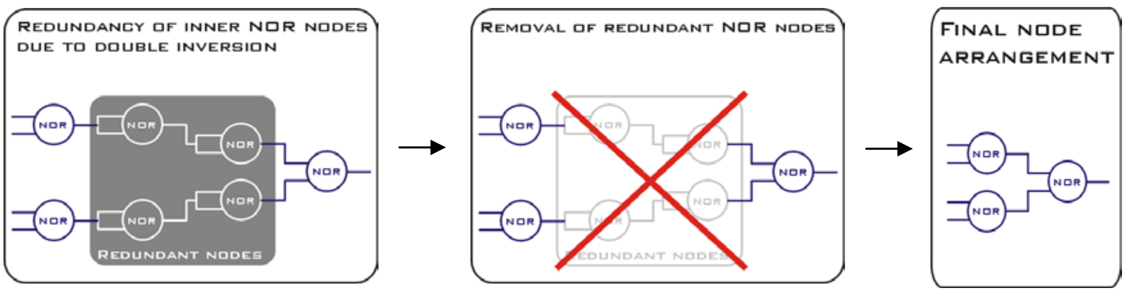
Although we could work with general MIN-MAX trees, in order to simplify the mathematics of our discussion we will focus our attention on so-called AND-OR trees. An AND-OR tree is a special type of binary MIN-MAX tree, where [AND] and [OR] nodes are MIN and MAX nodes, respectively. Possible values for tree leaves are, of course, binary "0" or "1".



The first step is a simple modification of an AND-OR tree, which replaces all [AND] and [OR] nodes by equivalent elements composed of [NOR] nodes (i.e. an [OR] is a negated [NOR] etc).


"OR" AND "AND" NODES CONSTRUCTED FROM "NOR" NODES
OR
NOR
NOR
AND
NOR
NOR
NOR
SINGLE MIN-MAX "BUILDING BLOCK"
OR
OR
AND
OR
NOR
NOR
OR
NOR
NOR
AND
NOR
NOR
NOR

This yields a NOR tree (which has the same depth due to redundancy of some of the inner nodes). Though identical in terms of functionality, it also possesses a completely homogenous structure, which further facilitates our calculations.


REDUNDANCY OF INNER NOR NODES DUE TO DOUBLE INVERSION
NOR
NOR
NOR
NOR
NOR
NOR
NOR
REDUNDANT NODES
REMOVAL OF REDUNDANT NOR NODES
NOR
NOR
NOR
FINAL NODE ARRANGEMENT
NOR
NOR
NOR

Second, we need to specify a probability distribution on values of the leaves. Each leaf is independently set to "1", with the following probability:

$$\boxed{p = \frac{3-\sqrt{5}}{2}}$$

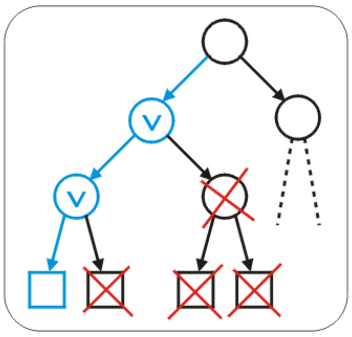
(Note: our peculiar choice of p will be justified by the next equation.)

The probability of a [NOR] node's output being "1" is the probability that both inputs are "0", i.e.:

$$\boxed{(1-p)(1-p) = \left(\frac{2}{2} - \frac{3-\sqrt{5}}{2}\right) * \left(\frac{2-3-\sqrt{5}}{2}\right) = \left(\frac{\sqrt{5}-1}{2}\right)^2 = \frac{3-\sqrt{5}}{2} = p}$$

Now we consider some properties of a deterministic algorithm evaluating our tree. Since we would like to minimize its running time, we make use of the following observation:

For a [NOR] node, the output value will be “0” if one of its children returns “1”. In such cases, the other child (possibly an entire sub-tree) will not influence the result, and therefore, doesn’t need to be inspected.
This leads to a notion of Depth-First Pruning algorithms (DFPA). A DFPA essentially functions like a DFS, but it also stops visiting sub-trees of a node once its value has been determined – sub-trees that yield no additional information are “pruned” away.



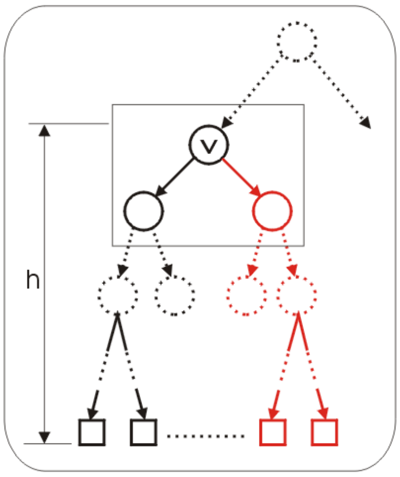
Observe the following proposition:

*Let T be a NOR tree, with all leaves set to the aforementioned distribution. Let $W(T)$ denote a minimum, over all deterministic game tree algorithms, of the expected number of steps to evaluate T. Then, there exists a DFP algorithm, whose expected number of steps to evaluate T is $W(T)$.*

Thus, for the purposes of our discussion, we may restrict ourselves to DFPAs.
For a DFPA, traversing a NOR tree with *n* leaves and aforementioned probability distribution, the following holds:

Let *h* be the distance from the leaves to the node in question. Let $W(h)$ denote the expected number of leaves the DFPA will need to inspect in order to evaluate the node.
Then:

$$\boxed{W(h) = W(h-1) + (1-p)\cdot W(h-1)}$$



Here $W(h-1)$ is the expected number of leaves visited while evaluating one of the sub-trees of the node. The factor $(1-p)$ before the second term arises from the fact that the other sub-tree will *only* be visited if the first sub-tree yielded 0, which will happen with the probability of $(1-p)$.

(Note: there is no factor p before the first term, as one might expect, because one of the sub-trees *must* be visited under all circumstances, i.e. with 100 percent probability.)

Now we let $\boldsymbol{h = \log_2 n}$ (since we are working with a binary tree), and substitute it into the above equation. The solution of this equation produces the following result:

$$\boxed{W(h) \geq n^{0.694}}$$

We have thereby proven the following theorem:

*The expected running time of any randomized algorithm that always evaluates an instance of a binary MIN-MAX tree correctly is at least* $n^{0.694}$*, where* $n=2^k$ *is the number of leaves.*

Our result is slightly less than the bound of $n^{0.793}$ presented in the previous discussion by Alexander Hombach. However, our method is correct (since it is based on Yao's Technique). One possibility is that our distribution of input values is not optimal, since it does not preclude the possibility of both inputs to a NOR node being "1". A distribution that prevents such a possibility would show that the evaluation algorithm introduced in the previous presentation is indeed optimal.

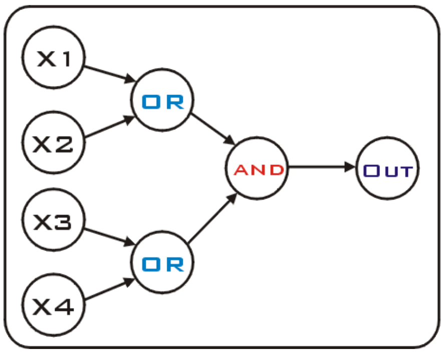
## Randomness and Non-Uniformity

In the second part of our discussion, we try to answer the following question:

*When is it possible to remove randomization from a randomized algorithm?*

For our analysis we need to introduce the notion of a randomized circuit. First, we give the definition of a *Boolean* circuit:

A *Boolean* circuit with n inputs is a DAG with following properties:



- It has *n* input vertices of in-degree 0, labeled $x_1, x_2, \ldots\ldots x_n$

- It has one output vertex of out-degree 0.

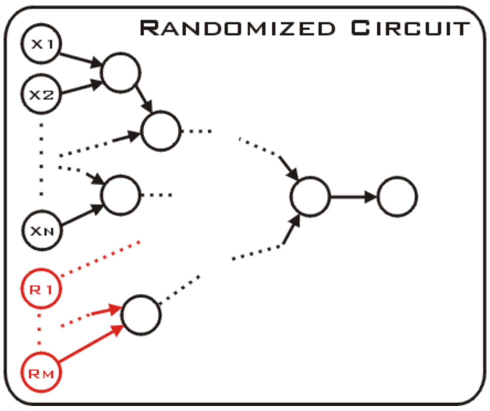
- Every inner vertex is labeled with a Boolean function from the set **[AND, OR, NOT]**. A vertex labeled **[NOT]** has in-degree 1.

- Every input can be assigned either 0 or 1.

- The output is a Boolean function of $x_1, x_2, \ldots\ldots x_n$. The circuit is said to compute this function.

- The size of the circuit is the number of vertices in it.

A *randomized* circuit is very similar to a Boolean circuit in terms of vertex properties, but in addition to n circuit inputs it also has several random inputs, labeled $r_1, r_2, \ldots\ldots r_n$. It computes a function of $x_1, x_2, \ldots\ldots x_n$ if following conditions hold:



- For all $x_1, x_2, \ldots\ldots x_n$ with $f(x_1, \ldots., x_n) = 0$ the output of the circuit is 0, regardless of the values of random inputs.
- If $f(x_1, \ldots., x_n) = 1$, the output is 1 with a probability $p \geq \frac{1}{2}$.

Now consider a Boolean function $f : \{0,1\}^* \to \{0,1\}$.
Let $f_n$ denote the function $f$ restricted to inputs from $\{0,1\}^n$. A sequence $C = C_1, C_2, \ldots$ is called a *circuit* family for $f$ if $C_n$ has $n$ inputs and computes $f_n(x_1, \ldots., x_n)$ for all n-bit inputs $(x_1, x_2, \ldots\ldots x_n)$. The family $C$ is polynomial-sized if the size of $C_n$ is bounded by a polynomial in $n$ for $\forall n$.

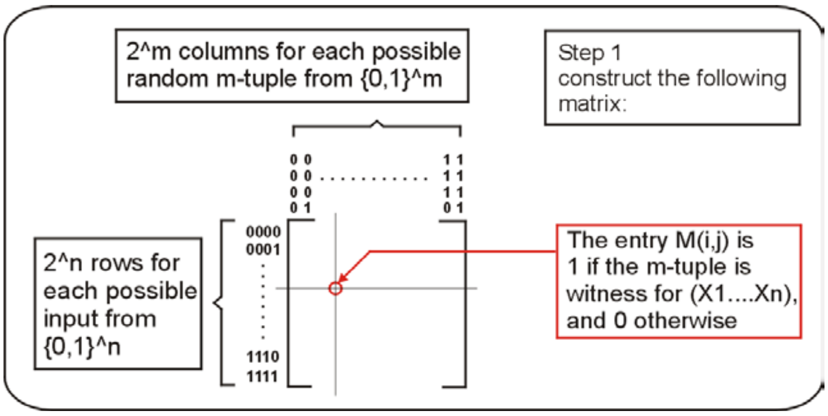
A *randomized* circuit family for $f$ is a family of randomized circuits, which has $m$ random inputs $r_1, r_2, \ldots\ldots r_m$ in addition to inputs $x_1, x_2, \ldots\ldots x_n$, with $r_1, r_2, \ldots\ldots r_m$ being either 0 or 1 with equal probability. The properties of the circuits concerning random inputs are those defined above.
All m-tuples $(r_1, r_2, \ldots\ldots r_m)$, for which $f_n(x_1, \ldots., x_n) = 1$ for a particular n-tuple $(x_1, x_2, \ldots\ldots x_n)$, are referred to as "witnesses" - they "testify" to the correct value of $f_n(x_1, \ldots., x_n) = 1$.

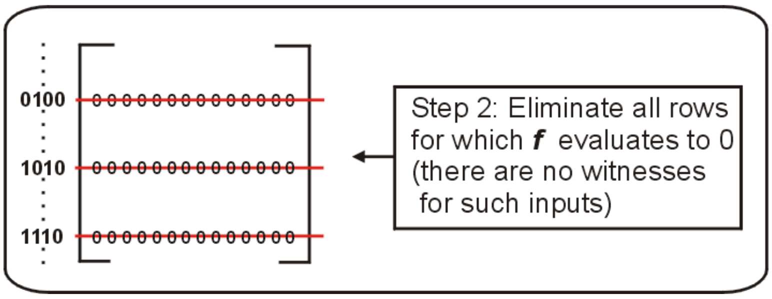
We now introduce **Adleman's Theorem**:

*If a Boolean function has a randomized, polynomial-sized circuit family, then it has a polynomial-sized circuit family.*

As a proof we provide a method that removes randomization from a randomized polynomial-sized circuit $C_n$ for $f_n(x_1, \ldots., x_n, r_1, \ldots. r_m)$ and transforms it into a deterministic polynomial-sized circuit $D_n$ that computes $f_n(x_1, \ldots., x_n)$:

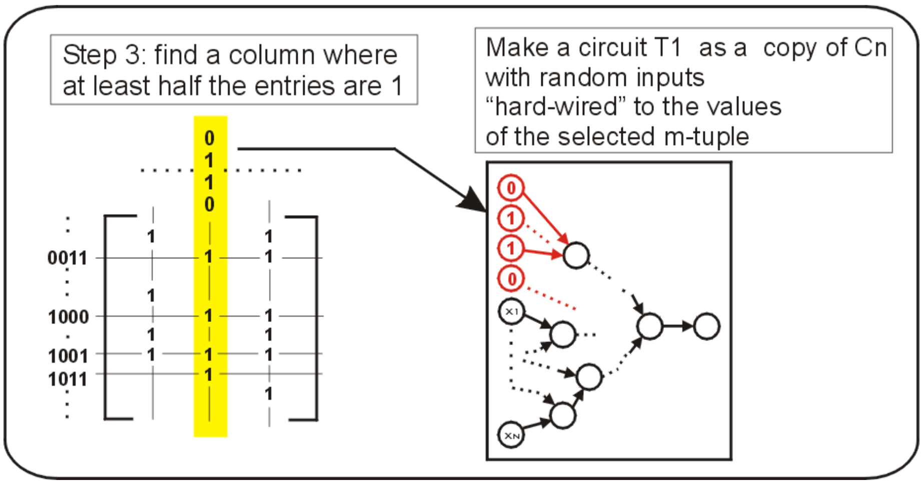
First, we construct a matrix M with $2^n$ rows for each possible n-tuple from $\{0,1\}^n$ and $2^m$ columns for each possible random m-tuple from $\{0,1\}^m$. An entry $M_{ij}$ is 1 if the corresponding m-tuple is witness for $(x_1,....,x_n)$, and 0 otherwise. Next, we eliminate all rows for which $f$ evaluates to 0, as there are no witnesses for such inputs.


2^m columns for each possible random m-tuple from {0,1}^m
Step 1 construct the following matrix:
2^n rows for each possible input from {0,1}^n
0000
0001
1110
1111
The entry M(i,j) is 1 if the m-tuple is witness for (X1....Xn), and 0 otherwise

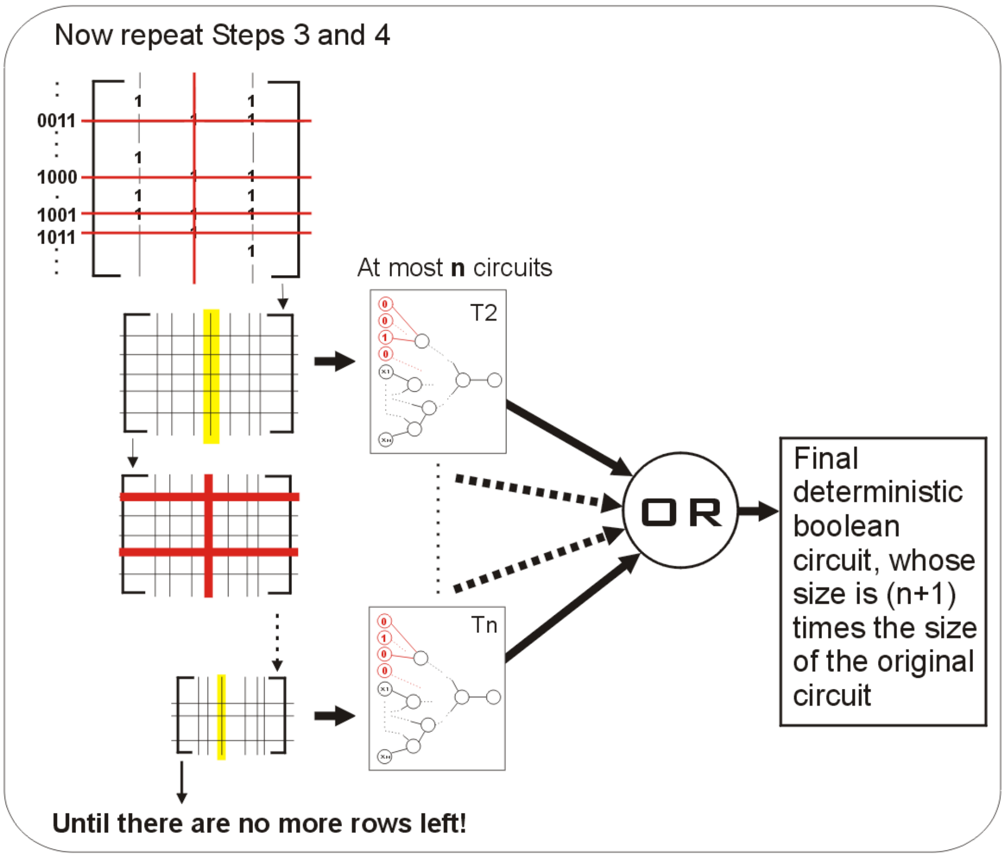

0100
1010
1110
Step 2: Eliminate all rows for which f evaluates to 0 (there are no witnesses for such inputs)

We start the construction of our circuit by finding a column in which at least half the entries are 1, that is, $f_n(x_1,....,x_n,r_1,....r_m)=1$ for at least half the possible inputs $(x_1,....,x_n)$.

We then construct a circuit $T_1$ as a copy of $C_n$ with random inputs "hard-wired" to the values of the selected m-tuple (note that such a circuit is purely deterministic!), and decimate the matrix by eliminating the selected column and all the rows that had 1's in it.


Step 3: find a column where at least half the entries are 1
Make a circuit T1 as a copy of Cn with random inputs "hard-wired" to the values of the selected m-tuple
0011
1000
1001
1011

Now we proceed in a similar fashion by selecting another column etc., until there are no more rows left. As a result, we will have constructed at most n circuits $T_1, T_2, \ldots T_n$, which we then combine into the final deterministic Boolean circuit, whose size is $(n+1)$ times the size of the original randomized circuit.


Now repeat Steps 3 and 4
0011
1000
1001
1011
At most **n** circuits
T2
Tn
OR
Final deterministic boolean circuit, whose size is (n+1) times the size of the original circuit
Until there are no more rows left!

Our method is an example of a *derandomization* technique. Derandomization often proves useful in design of deterministic algorithms – sometimes it is easier to devise a randomized algorithm as a solution to some problem, and then derandomize it to arrive at a deterministic algorithm. Unfortunately, it is not always possible or feasible to remove randomization from polynomial-time computations, due to the issue of *non-uniformity* in algorithms.

For further discussion we need to know what can be considered a non-uniform (or a uniform) algorithm:

Let $L$ denote a language over an alphabet $\sum^*$, and $a : IN \to \sum^*, n \to a(n)$ be a mapping from positive integers to strings in $L$. An algorithm $A$ is said to use the ***advice*** $a$ if on an input of length $n$ it is given a string $a(n)$ on a read-only tape.
$A$ decides $L$ with $a$ if on an input $x$ it uses $a(|x|)$ to decide $x \in L$. In other words, a single $a(n)$ enables $A$ to decide whether or not $x \in L$ for $\forall x, |x| = n$.

A *uniform* algorithm is an algorithm that doesn't use such advice strings at all.
A *non-uniform* algorithm utilizes such advice strings.

For the complexity class $P$ we define the class $P/poly$ as a class of all languages $L$ that have a non-uniform polynomial-time algorithm $A$, such that length of all advice strings $a(n)$ is polynomial-bounded in $n$, i.e. $|a(n)| = O(poly(n))$. Likewise, we may define the class $RP/poly$.

As an example, imagine a non-uniform algorithm $A$ that processes words $\{x \in \sum^*, |x| = n\}$. Let $a(n)$ contain all $\{x \in L, |x| = n\}$. $L$ would be in $P/poly$ if the total number of words in $L$ were bounded by $poly(n)$.

Similarly, we may speak of a language L as having a randomized circuit family. Then, $L \in RP/poly$ if and only if it has a randomized polynomial-sized circuit family.
Hence, one may interpret Adleman's Theorem as a proof that

$$RP/poly \subseteq P/poly$$

However, this only shows that removal of randomization can be done in principle. There exist no uniform or practical methods for achieving this.

## SUMMARY

In this discussion, we have covered the topics of randomized game tree algorithms, Minimax Principle and Von Neumann's Theorem, as well as Yao's Techinques as powerful tools for bound estimation. We also presented a method for evaluating the lowest possible bound for a randomized algorithm, and addressed the issues of randomization removal and non-uniformity in algorithms.

Bibliography:

Randomized Algorithms" by Rajeev Motwani and Prabhakar Raghavan
Cambridge University Press, 1995