MATHEMATICAL FOUNDATIONS OF DISCRETE GEOMETRY.

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ABSTRACT

Nowadays Discrete Geometry means two different things.

For the Mathematician it is the study of Packings and Coverings, (cf. Rogers –*Packing and Covering*– Cambridge University Text, 1964, and Conway-Sloane –*Sphere Packings, Lattices and Groups*– Springer, 1988). For the Computer Scientist, it is the gathering of properties of usual euclidean discretizations: lines, circles, spheres... This shows the need for a Geometry of discrete structures, such as $\mathbb{Z}^n$ lattices for example, in order to solve problems encountered with today's numerous digitalization devices, (cf Rosenfeld-Kak –*Digital Image Processing*– Academic Press 1982, Chassery-Montanvert –*Géométrie Discrète en Analyse d'Images*– Hermès 1991). This theory seems also much wanted in other fields such as Solid State Physics or Cristallography.

If there is no foundational problem in the domain described by the Mathematician, that is not the case for Computer Scientist's one. This is exactly the subject this paper is concerned with: to give Mathematical Foundations of a Geometry on $\mathbb{Z}^n$ lattices.

In this lecture we restrict ourselves to the first non-trivial case of dimension n=2.

We will explain how this theory comes from the fact that we require it to satisfy the five following conditions:

It is the Geometry of a discrete group: $Sl(2,\mathbb{Z})$,

its Algorithmics is as simple as possible,

its application is universal,

it has a Computational counterpart,

Euclidean Geometry can be recovered "in the large".

We will study these five points in turn.

1 Sl(2,$\mathbb{Z}$)'s part in Discrete Geometry.

We begin by introducing the following notion of discrete line:

DEFINITION. – *A discrete rational line $\mathcal{D}$ is the set of integer solutions* (x,y) *of inequalities*

$$\gamma \leq ax+by < \gamma+\tau,$$

where a, b, γ, τ are integers and $\tau > 0$; we denote it by $\mathcal{D}(a,b,\gamma,\tau)$.

Here (a,b) is $\mathcal{D}$'s normal vector, values γ and $\gamma+\tau$ are $\mathcal{D}$'s bounds and positive integer τ is $\mathcal{D}$'s arithmetical thickness.

We will show that these lines possess interesting symetries and can be classified with respect to two kinds of transformations: plane integer translations and action of the Sl(2,$\mathbb{Z}$) group. The results are two main structure theorems which give, from a theoretical point of view, a "Kleinian" interpretation of this bidimensional Discrete Geometry.

2 Discrete Geometry's algorithmics.

Our approach offers a nice "abstract" point of view for treating Computer Scientists' everyday problems; this formalism is not akward at all. On the contrary it is of great help when designing algorithms concerning not only discrete lines, plane discrete transforms, but also several other questions as antialiassing, Moiré patterns, pattern analysis, quasi-crystals, fractals...

This part is mainly devoted to arithmetics and algorithmics of lines. More precisely we will show how modular sequences, Euclidean algorithm and continued fractions come interact to form a very interesting algorithmical domain concerning discrete lines. We also remind the reader that these questions come from very old concerns dating from Jean Bernoulli and E.C. Christoffel.

3 Universality of Discrete Geometry.

By universality we mean that a very great number of practical applications can be made by using a small number of abstract principles.

In Discrete Geometry, information is simultaneously geometrical and arithmetical. This coexistence sometimes creates much more subtle networks than we are accustomed to with Euclidean Geometry. Thus discrete notions interactions are completely different from euclidean ones. Several examples of this important fact are given.

Particularly the main part played in Discrete Geometry by lines' arithmetical thickness and intersection is explained and illustrated with the help of Discrete affine transforms, Moiré patterns, and Quasi-crystal structures.

The simple notion of discrete lines thickness is shown to have a far reaching usefulness. For example it allows bending and stretching of lines to directly control discrete metrical properties.

The intersection of discrete lines, which amounts to solve the following system of 4 diophantine inequations:

$$\gamma \leq ax+by < \gamma+\omega$$
$$\eta \leq cx+dy < \eta+\rho$$

also plays, in Discrete Geometry, a most important part.

Except in seldom cases, the problem of describing the integer points contained in two strips, has nothing to do with obtaining the integer part of the intersection point of two real lines. This is also a very old problem already studied, long before computer's birth, by Mac-Mahon, Farkas, Van der Corput... but with very different concerns.

In the discrete case, intersections may be void or may also contain an infinite number of points, (parallel and overlapping, though distinct, discrete lines exist); they also may be non-connected (in the sense of 4 or 8-connectivity).

We have two ways of solving this problem. The first one uses $Sl(2,\mathbb{Z})$'s structure and reduces the problem to a walk through a discrete segment. This result shows that intersection's complexity does not increase too quickly and remains equivalent to real lines case. A second approach, using lattices, will be briefly sketched.

Noting that our method recalls particular techniques of Numerical Matrix Analysis, we will present the new notion of Quasi Affine Transform (QATs for short), a diophantine analogue of linear affine transforms, and compare structures of both objects. These QATs possess dynamics which behave between two extreme situations: from "real-like" case, (one fixed point), to permutation (only cycles).

We can give an arithmetical characterization of the "real-like" case. We also give some of the QAT's general properties (cycles, leaves, points' inverse image, linear bidimensional reciprocity formula...) and applications (plane affine discrete transforms, antialiassing, fractals...).

This rather lengthy part ends with the fondamental inverse problem of recognizing discrete lines. The characterization we obtained is used to give a linear and very simple algorithm for decomposing a discrete curve into discrete line segments.

4 An Analytical approach of Discrete Geometry.

It is well know that Euclidean Geometry is equivalent to Analytical Geometry; in this part we tackle the difficult question of Discrete Geometry's Analytical counterpart. The answer to this question still remains unclear; we explain what this Discrete Analytical Geometry should be: a Calculus about the Integer Part Function already wanted by F. Gauss.

Though we do not have yet a general answer, numerous, scarce, but related, elements are explained:

Linear reciprocity, parametrizations, quadratic reciprocity, Ramanujan's lattice problem, Smith normal form for integer matrices, Jacobi's Theta series...

5 How to recover Euclidean Geometry from Discrete Geometry.

In this last part an Ideal Discrete Geometry, infinitely close to the Euclidean Geometry, is built and connected to the Discrete Geometry just presented. A simple explanation of this construction requires the most elementary notion of Non-Standard-Analysis: non-standard integers. We use it to introduce a theoretical non-standard discrete screen (infinitely large) and show that its Geometry is a lifting of Euclidean Geometry which can axiomatically be treated. Moreover this theory, which can be connected to the Discrete Geometry studied in the preceeding parts, can be considered as its limit "in the large".

This is what we mean by a unified Mathematical treatment of Discrete Geometry.