# The complexity of the min-max degree triangulation problem*

Klaus Jansen
Fachbereich IV, Mathematik und Informatik,
Universität Trier, Postfach 3825, W-5500 Trier, Germany

November 23, 1992

Let $V$ be a set of $n$ points in $\mathbb{R}^2$. An *edge* is a closed line segment connecting two points of $V$. Let $E$ be a set of edges. Then $G = (V, E)$ is a *geometric graph* if for every edge $ab \in E$, $ab \cap V = \{a, b\}$. A geometric graph is called *plane* if for every two edges $ab \neq cd$ in $E$, either $ab \cap cd = \emptyset$ or $ab \cap cd$ is an endpoint of both edges.

The connected components of $\mathbb{R}^2$ minus all points in $V$ and on edges of $E$ are the *faces* of $G$. If the edges in $E$ are pairwise disjoint, then $G$ is a matching and we have only one unbounded face. If $V$ is fixed and $E$ is maximal such that no two edges cross, then $G$ is a *geometric triangulation* of the convex hull of $V$. Then, the bounded faces of a triangulation are triangles. A *triangulation* of a geometric graph $G = (V, E)$ is a plane geometric graph $G' = (V, E')$, where $E \subseteq E'$ and where $G'$ is a geometric triangulation. The *degree* $\delta(v)$ of a point $v$ of a graph is the number of incident edges. Let $\Delta(G)$ be the maximum degree over all points in $G = (V, E)$.

The problem studied here is described as follows. Let $G = (V, E)$ be a plane geometric graph. The problem is to find a triangulation $G'$ of $G$ that minimizes $\Delta(G')$. Clearly, for a triangulation of this form it is not allowed to add new points. The decision problem has the following form:

**Problem:** Min-max degree triangulation

**Given:** A plane geometric graph $G = (V, E)$ in $\mathbb{R}^2$ with finite point set $V$ and edge set $E$, and an integer $k \in \mathbb{N}$.

**Question:** Is there a triangulation $G'$ of $G$ with maximum degree $\Delta(G') \leq k$ ?

This problem was raised as an open problem by Herbert Edelsbrunner [4]. We note that for any triangulation $G'$ of a plane geometric graph $G = (V, E)$ with $|V| \geq 5$, $\Delta(G') \geq 4$ holds and that for any integer $n \geq 5$ there exist triangulated graphs $G' = (V, E')$ with $|V| = n$ and $\Delta(G') = 4$. An application of our studied problem in numerical engineering is given by Frey and Field [5].

The problem to find a triangulation $\overline{G} = (V, \overline{E})$ as a subset of a geometric graph $G = (V, E)$ with $\overline{E} \subseteq E$ is studied by Lloyd [9]. Using a reduction from 3-SAT he showed

*Research of the author was supported by the Deutsche Forschungsgemeinschaft and was done a visit at the University of Illinois at Urbana-Champaign

that this triangulation problem is NP-complete. Given a plane geometric graph with or without constraining edges, several optimal triangulation problems have been studied [1, 2, 3]. Optimal means that the form of the triangles or the triangulations is optimized. In contrast to polynomial algorithms in [1, 2, 3], we give the first negative result for an optimal triangulation problem.

The NP-completeness of a similar problem to triangulate a planar graph while minimizing the maximum degree has been proved by Kant and Bodlaender [7]. One difference in [7] to our considered problem and to the studied triangulation problems in [1, 2, 3] is that an embedding of the graph in the plane is not given in the problem instance. The second important difference is that the constructed lines in the triangulation in [7] are not straight lines. Our main result is the following.

**Theorem.** The min-max degree triangulation is NP-complete for $k = 7$.

**Proof.** By reduction from a restricted version of planar 3-SAT which is also NP-complete [8]:

**Problem:** Restricted planar 3-SAT

**Given:** A formula $\phi$ with a set $C$ of clauses over a set $X$ of variables that satisfies the following three conditions.

(i) Each clause contains at most three and at least two literals.

(ii) Each variable occurs in at most three and at least two clauses, where we count $x$ as well as $\overline{x}$ as an occurrence of $x \in X$.

(iii) The undirected graph $G_\phi$ is planar.

**Question:** Is $\phi$ satisfiable ?

Now we give an overview of the reduction. Given a planar graph $G_\phi$ for a formula $\phi$ we compute in the first step of the reduction a rectilinear planar layout with horizontal lines for vertices and vertical lines for edges. Then we grow the lines to rectangles of unit height or width. An example of a planar graph, its rectilinear layout and its modified layout is given in Figures 1 and 2.

In the second step we construct blocks that represent variables, complemented variables and clauses. These are placed inside the corresponding horizontal rectangles. For each variable $x \in X$ we generate a block that allows us to assign a truth value by choosing a triangulation of the block. To get a complemented variable we use an inverter. For each clause $c \in C$ we generate a block that can be triangulated with maximum degree seven or less if and only if at least one of the corresponding literals has the truth value true. For moving the information between a literal and a clause we use a sequence of squares.

We simplify the construction in the second step and give only a part of each block with points of small degree. To understand the function of the block designs, we assume that the used points in the blocks have a larger degree five or six. To get these degrees for the points, we add in the last step of the reduction for each block some new points connected to the old ones. Furthermore, the new points allow us to triangulate the regions between the block designs with degree less than eight.

The block designs for a connection, a variable, an inverter and a clause are given in Figures 3, 5, 8 and 10. Some of choices for a triangulation of the block designs are illustrated in Figures 4, 6, 7, 9 and 11. We omit the details in the last step to generate a degree of five or six at the points in the block designs.

We have proved that the triangulation problem is NP-complete with maximum degree seven. Using a more complicated construction it might be possible to improve this to degree six. The complexity of the min-max degree problem without any constraining edges remains open.

Another problem is to find a triangulation of a point set with minimizing the sum of the edge distances. The complexity of this problem called the minimum length triangulation problem is one of the remaining open problems of Garey and Johnson [6]. Surprisingly, the triangulation problem considered in this paper is NP-complete, which gives hope that some of the ideas will eventually lead to an NP-completeness proof for the minimum length triangulation problem.

We thank Herbert Edelsbrunner for directing us to the min-max degree triangulation problem and for improving the paper. Moreover, we thank all members of the research group of Herbert Edelsbrunner in Urbana Champaign for helpful discussions.

## References

[1] M. Bern, H. Edelsbrunner, D. Eppstein, S. Mitchell and T.S. Tan, Edge insertion for optimal triangulations, Proc. Latin America Theoretical Informat. (1992).

[2] H. Edelsbrunner, T.S. Tan and R. Waupotitsch, An $O(n^2 logn)$ time algorithm for the minmax angle triangulation, Proc. 6th Ann. Sympos. Comput. Geom. (1990), pp. 44 – 52.

[3] H. Edelsbrunner and T.S. Tan, A quadratic time algorithm for the minmax length triangulation, Proc. 32nd Ann. Sympos. Found. Comp. Sci. (1991), pp. 414 – 423.

[4] H. Edelsbrunner, Triangulation lecture notes, University of Illinois at Urbana Champaign (1991).

[5] W.H. Frey and D.A. Field, Mesh relaxation: a new technique for improving triangulations, International Journal for numerical methods in engineering, 31 (1991), pp. 1121 – 1133.

[6] M.R. Garey and D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-completeness, Freeman, San Francisco (1979).

[7] G. Kant and H.L. Bodlaender, Triangulating planar graphs while minimizing the maximum degree, Proc. 3rd Sca. Workshop Algorithm Theory, LNCS **621** (1992), pp. 258 – 271.

[8] D. Lichtenstein, Planar formulae and their uses, SIAM J. Comput. 11 (1982) pp. 320 – 343.

[9] E.L. Lloyd, On triangulations of a set of points in the plane, Proc. 18th Ann. IEEE Sympos. Found. Comput. Sci. (1977), pp. 228 – 240.

[10] P. Rosenstiehl, R.E. Tarjan, Rectilinear planar layouts of planar graphs and bipolar orientations, Disc. Comput. Geom. 1 (1986) pp. 343 – 353.

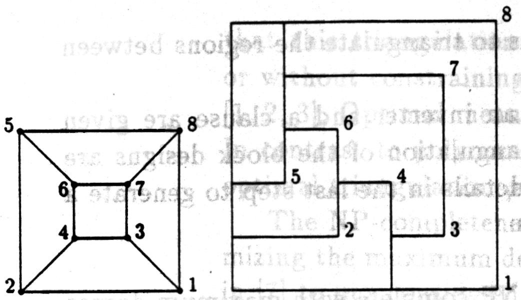


Figure 1: A planar graph and its rectilinear layout

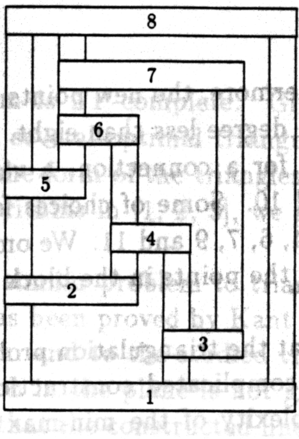


Figure 2: The modified layout with rectangles

Figure 3: A path of squares in a vertical rectangle

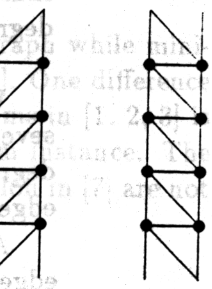

Figure 4: Two ways to triangulate a path of squares

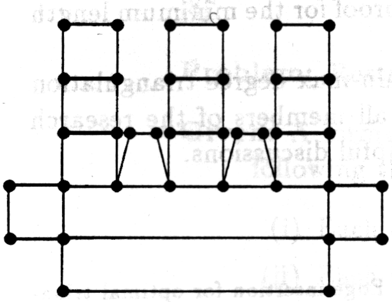

Figure 5: The variable setting

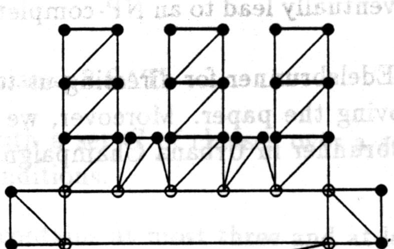

Figure 6: The first choice

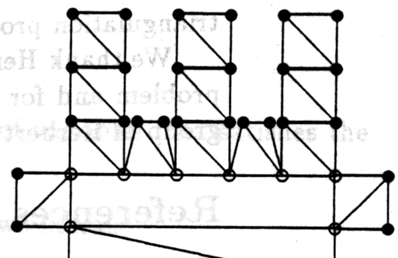

Figure 7: The second choice

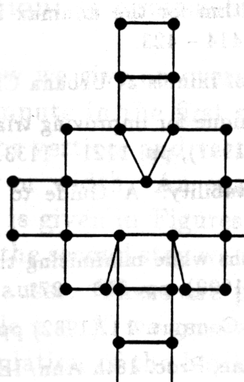

Figure 8: The block design for an inverter

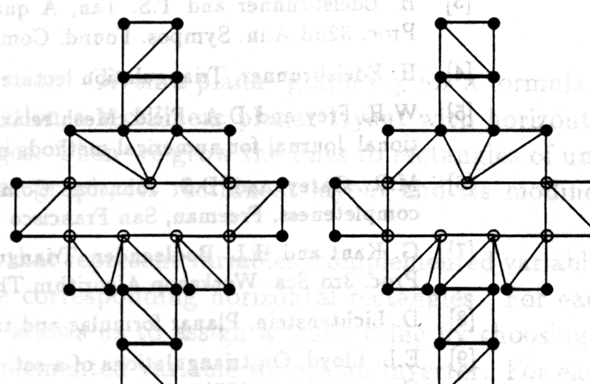

Figure 9: Two signal transformations in an inverter

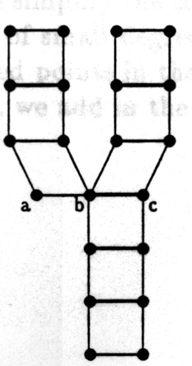


Figure 10: One construction for a clause with three literals

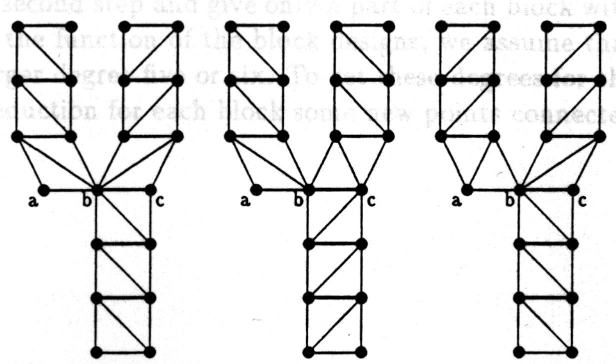


Figure 11: Three of eight configurations for a clause of the first form