Triangulations, Visibility Graph and Reflex Vertices of A Simple Polygon

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Abstract. In this paper tight lower and upper bounds for the number of triangulations of a simple polygon are obtained as a function of the number of reflex vertices it has, so relating these two shape descriptors. Tight bounds for the size of the visibility graph of the polygon are obtained too, with the same parameter. The former bounds are also studied from an asymptotical point of view.

Keywords. Simple polygon, triangulation, decomposition, visibility, visibility graph.

1. Introduction

The number of reflex vertices of a simple polygon is a shape complexity measure describing how far it is from the convex paradigm of simplicity. This number is a poor descriptor when considered alone, as pointed by Toussaint [9]: given any polygon, if we insert a new vertex R in some original side P_iP_{i+1} and pull it an infinitesimal amount towards the interior of the polygon, the basic shape will remain unchanged. In fact, only the visibility between P_i and P_{i+1} has been altered. But if R enters progressively in the interior of the polygon, the visibility between many pairs of vertices can disappear and R will become really significative.

The numbers of ways a polygon can be triangulated is again a shape descriptor. If the polygon has many arms, and it is very twisted, the number of triangulations will be relatively low, and this number will increase if there are important "convex bags", because many internal diagonals are then available.

An internal diagonal is a visibility trajectory, and could have been destroyed by a reflex vertex, so it is reasonable to expect a relation between the two numbers precedently considerated. Let n and k be the number of sides and reflex vertices of any polygon, respectively. Hertel and Mehlhorn [6] described an algorithm for triangulating a simple polygon that performs better the fewer reflex vertices it has (the running time is $O(n + k \log k)$). This is natural because in an average case, as we show in this paper, the fewer that number, the higher the number of possible triangulations.

The proof is based in decomposing the polygon in convex pieces, a subject widely studied, but the main objective is usually to minimize the number of pieces. Chazelle and Dobkin [1] [2] [3] obtained a running time upper bound $O(n + k^3)$ with Steiner points allowed. The algorithms by Greene [5] and Keil [7] use only vertices from the original set and have a worst case complexity $O(n^2k^2)$ and $O(k^2 n \log n)$, respectively. The survey [8] by Keil and Sack provides a panorama on the subject. In our paper a fixed number of non overlapping convex parts is sought, perhaps not covering all the polygon, but providing a joint size large enough for visibility purposes.

The size of the visibility graph of a polygon is naturally related to the number of reflex vertices it has, because adjacencies correspond to sides and internal diagonals, so it is not surprising the former decomposition provides bounds for that size too.

The paper is organized as follows. In Section 2 we establish the lemmae relating the number of triangulations of a polygon with the joint size of the "convex bags" it has, and this size with the number of reflex vertices, the results being combined together in a final theorem. In Section 3 precedent lemmae are used in relation with the visibility graph. Finally, in Section 4 we study from an asymptotical point of view the lower and upper bounds obtained for the number of triangulations.

All the polygons we consider being simple, the adjective will be omitted hereafter. The vertices will be numbered counterclockwise with indices modulus the number of sides.

2. The number of triangulations of a polygon

Let t_n and u_n be the number of triangulations of a convex n-polygon and the nth number of Catalan, respectively. It is a well known fact (see for example [4]) that these sequences are the same up to a shift of two positions: more precisely

$$t_n = u_{n-2} = \frac{1}{n-1}\binom{2n-4}{n-2} \qquad (n \geq 3).$$

We will accept as a convention that a segment is a convex polygon with two sides and $t_2 = u_0 = 1$.

Given a polygon P, we denote $t(P)$ the number of triangulations of P. The following theorem describes the general situation.

Theorem 1. *Let P be a n-polygon. Then $1 \leq t(P) \leq t_n$, and these bounds are tight.*

The presence of reflex vertices diminishes the internal visibility (at least between the vertices adjacent to the concave one), like that some triangulations are lost with regard to the convex model. But no necessarily the more concave vertices we have the less triangulations we can obtain.

The main theorem in this section provides tight bounds for the number of triangulations of a polygon as a function of its reflex vertices. To establish that result we need some lemmae. We use the Catalan numbers u_n for the purely combinatorial results, the rephrasing in terms of the t_n being identical.

Lemma 1. *Let $\alpha_1, \ldots, \alpha_m, w$ nonnegative integers with $\alpha_1 + \cdots + \alpha_m \geq w$. Then*

$$u_{\alpha_1} \cdots u_{\alpha_m} \geq \left(u_{\lceil \frac{w}{m} \rceil}\right)^t \left(u_{\lfloor \frac{w}{m} \rfloor}\right)^{m-t}$$

where t is the residue of the division of w by m.

If a polygon P admits a collection of non-overlapping convex subpolygons, the product of its corresponding numbers of triangulations is a lower bound for $t(P)$, and this value will be the lesser the more equilibrated the sizes of the "convex bags". Next lemmae will provide us a certain relation between the number of "bags" and its collective size.

Definitions. Let P be a polygon with vertices $P_0, \ldots, P_{n-1}$ and P_i a concave vertex of P. We say a triangle $P_sP_tP_i$ *breaks* the angle in P_i when its interior is contained in P and neither of the angles $P_{i+1}P_iP_s$, $P_sP_iP_t$, $P_tP_iP_{i-1}$ is concave. An internal diagonal P_iP_k *breaks* the angle in P_i if the angles $P_{i-1}P_iP_k$ and $P_kP_iP_{i+1}$ are not concave. We use the term "to break" because we intend to work with the resultant pieces. The *central zone* of P_i is the part of the plane to the right of the ray $P_{i+1}P_i$ and to the left of the ray $P_{i-1}P_i$.

The main strategy for the next results is as follows: to break the polygon in pieces by a diagonal or a triangle, diminishing the total number of reflex vertices, and then iterating the process. This can be done in such a way that we obtain the following result:

Lemma 2. *Let P be a n-polygon with k reflex vertices. There are $k+1$ convex polygons $C_1, \ldots, C_{k+1}$ such that*

a) Every vertex of C_i is a vertex of P, $i = 1, \ldots, k+1$;

b) $\sum_{i=1}^{k+1} |vertices(C_i)| \geq n + k$;

c) If $i \neq j$ then $C_i \cap C_j$ is empty, a common vertex or a common edge;

d) Points interior to C_i are interior to P, $i = 1, \ldots, k+1$.

Now we have all the ingredients for our main result:

Theorem 2. *Let P be a n-polygon with k reflex vertices. Then*

$$\left(t_{\lceil \frac{n+k}{k+1} \rceil}\right)^{s} \left(t_{\lfloor \frac{n+k}{k+1} \rfloor}\right)^{k+1-s} \leq t(P) \leq t_n - \binom{k}{1} t_{n-1} + \cdots + (-1)^k \binom{k}{k} t_{n-k},$$

where s is the residue of the division of $n+k$ by $k+1$, and these bounds are tight.

The precedent results have some simple consequences that are worth noting:

Observation 1. The lower bound in Theorem 2 is 1 when $2 \leq \frac{n+k}{k+1} \leq 3$ or, equivalently, $k \geq \frac{n-3}{2}$. So, if $k < \frac{n-3}{2}$ then P admits at least two triangulations.

Observation 2. A n-polygon with k concave vertices admits always a convex $\lceil \frac{n+k}{k+1} \rceil$-subpolygon, and this value is tight. In other words, in order to guarantee a convex p-subpolygon we must have $n \geq kp + p - 2k$.

3. The size of the visibility graph

Let $G_P = (V_P, E_P)$ be the visibility graph of a simple polygon P. Some results we have already given can be translated in terms of the visibility graph. For example, Observation 2 say us that if P has n sides and k reflex vertices then $clique(G_P) \geq \lceil \frac{n+k}{k+1} \rceil$, and the bound is tight. The size $|E_P|$ of G_P is also naturally related to the number of reflex vertices it has, because internal diagonals -adjacencies in G_P- can be destroyed. In this section we precise that relation.

Next theorem is the (trivial) statement for the general situation.

Theorem 3. *Let P be a n-polygon and $G_P = (V_P, E_P)$ its visibility graph. Then $2n - 3 \leq |E_P| \leq \binom{n}{2}$, and these bounds are tight.*

This bounds can be improved if we know the number of reflex vertices, by using the decomposition obtained in the precedent section.

Lemma 3. *Let $n_1, \ldots, n_m$ be nonnegative integers. Then*

$$\binom{n_1}{2} + \cdots + \binom{n_m}{2} \geq s\binom{\lceil \frac{\sum n_i}{m} \rceil}{2} + (m-s)\binom{\lfloor \frac{\sum n_i}{m} \rfloor}{2}$$

where s is the residue of the division of $\sum n_i$ by m.

Theorem 4. *Let P be a n-polygon with with k reflex vertices, and let $G_P = (V_P, E_P)$ its visibility graph. Then*

$$k + s\binom{\lceil \frac{n+k}{k+1} \rceil}{2} + (k+1-s)\binom{\lfloor \frac{n+k}{k+1} \rfloor}{2} \leq |E_P| \leq \binom{n}{2} - k$$

where s is the residue of the division of $n+k$ by $k+1$, and these bounds are tight.

4. Asymptotic analysis

We now proceed to compute asymptotic estimates for the bounds presented in Theorem 2. In order to simplify matters, we will content ourselves with the following expressions for the lower and upper bounds, which do not alter its asymptotic behaviour:

$$L_k(n) = (u_{\lfloor \frac{n}{k+1} \rfloor})^{k+1}$$

$$U_k(n) = u_n - \binom{k}{1} u_{n-1} + \binom{k}{2} u_{n-2} + \cdots + (-1)^k \binom{k}{k} u_{n-k}$$

Theorem 5. *Let $L_k(n)$ and $U_k(n)$ be as above. Then*

$$L_k(n) \sim \rho\, 4^n n^{-3(k+1)/2} \text{ for a certain constant } \rho = \rho(k);$$
$$U_k(n) \sim (\frac{3}{4})^k \pi^{-1/2} 4^n n^{-3/2}.$$

Thus the main asymptotic term 4^n in $L_k(n)$ remains the same as in the convex case ($k = 0$) but the degree of the polynomial in n has been decreased by $3k/2$. The result for $U_k(n)$ can be rephrased in a simple way as

$$\frac{U_k(n)}{u_n} \sim (3/4)^k;$$

that is, every time we add a reflex vertex, the maximum number of triangulations of a polygon is decreased (asymptotically) by a factor of 3/4.

Observation 3. As an additional remark, we note that if make the number of reflex vertices is proportional to n, say $k = n/\alpha$, then

$$L_k(n) = ((u_\alpha)^{\frac{1}{\alpha}})^n.$$

The sequence $(u_\alpha)^{\frac{1}{\alpha}}$ is increasing with limit equal to 4 (this follows from the fact that $u_{\alpha+1}/u_\alpha = 2\,\frac{2\alpha+1}{\alpha+2}$ has limit 4 as α goes to infinity). This means that we are always asympotatically under the main term 4^n coming from the convex case, but we approach this limit as the number of reflex vertices becomes relatively scarce. As for the upper bound, similar considerations apply from its asymptotic estimate.

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