POLYTOPE CONTAINMENT AND DETERMINATION BY LINEAR PROBES

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ABSTRACT

As the terms are used here, a *body* in $\mathbb{R}^d$ is a compact convex set with nonempty interior, and a *polytope* is a body that has only finitely many extreme points. The class of all bodies whose interior includes the origin 0 is denoted by $\mathcal{C}_0^d$. A set X is *symmetric* if $X = -X$. The *ray-oracle* of a body $C \in \mathcal{C}_0^d$ is the function $\mathcal{O}_C$ which, accepting as input an arbitrary ray R issuing from 0, produces the point at which R intersects C's boundary. This paper is concerned with a few central aspects of the following general question: Given certain information about C, what additional information can be obtained by questioning the ray-oracle, and how efficiently can it be obtained? It is assumed that the usual arithmetic operations in $\mathbb{R}^d$ are available at no cost, so the efficiency of an algorithm is measured solely in terms of its number of calls to the ray-oracle.

The paper discusses two main problems, the first of which – the *containment problem* – arose from a question in abstract numerical analysis. Here the goal is to *construct* a polytope P (not necessarily in any sense a small one) that contains C, where this requires precise specification of P's vertices. There are some sharp positive results for the case in which $d = 2$ and C is known not to be too asymmetric, but the main result on the containment problem is negative. It asserts that when $d \geq 3$ and the body is known only to be rotund and symmetric, there is no algorithm for the containment problem. This is the case even when there is available a certain master oracle whose question-answering power is much greater than that of C's ray-oracle. However, it turns out that even when there is no additional information about C, the following relaxation of the containment problem admits

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an algorithmic solution based solely on the ray-oracle: Construct a polytope containing C or conclude that C's centered condition number exceeds a certain bound.

In the other main problem -- the *reconstruction problem* – it is known only that C is itself a polytope and the problem is to construct C with the aid of a finite number of calls to the ray-oracle. That is accomplished with a number of calls that depends on the combinatoral complexity of C.