On Vertices and Polyhedra without Minimal Boolean Formulæ

Extended Abstract

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1. Introduction

The conversion of a solid represented by its boundary (boundary representation, B-rep) to a Boolean or set theoretic representation (constructive solid geometry, CSG) is one of the main problems of solid modelling [RV83]. For simple polygons in 2D, this problem has been solved [DG88]. For 3D polyhedra, on the other hand, only trivial bounds are known for the conversion time or the size of the result.

The most attractive feature of the 2D solution is the minimality of the size of the result. That a minimal Boolean formula (MBF) cannot be obtained for all 3D polyhedra has been demonstrated elsewhere [DG88],[DK89]. Here we show, by giving an example, that this is not possible even for the restricted case of a single vertex of a simplicial polyhedron, or, what is equivalent, for a simplicial polygon on the sphere.

The next section gives some details about the terms used in this paper and especially discusses the relationship between polyhedra, vertices, and spherical polygons. Section 3 then reviews the previously known cases of polyhedra without MBF. The new example without MBF is presented in Section 4.

2. Polyhedra, Vertices, and MBF

A *polyhedron* is, as usual, a connected subset of 3D Euclidean space bounded by non-intersecting simple planar polygons called faces. Two polygons meet at an edge, and several polygons and edges meet at a vertex.

The term *vertex* is generally only used for a single point; here, for convenience, it also denotes a second, related concept: the radial neighbourhood of such a point. This neighbourhood is obtained by removing faces not connected to the vertex and extending the remaining edges and faces to infinity. In a broader framework, a vertex can itself be understood as a polyhedron [Dü91].

Centring a sphere at the vertex point and intersecting it with the vertex, we obtain a *spherical polygon*. Each face of the vertex is mapped to an arc on a great circle of the sphere, and inside and outside of the vertex are mapped to inside and outside of the spherical polygon. The infinite radial extension of the vertex is collapsed, while fully conserving its structure.

A *simplicial* polyhedron is a polyhedron where all faces are triangles. A simplicial vertex is a vertex that is, or could be, part of a simplicial polyhedron. This is equivalent to saying that the angles of the faces meeting at the vertex are all convex. A simplicial spherical polygon then is a spherical polygon where all sides are shorter than 180°.

A Boolean or set theoretic formula for a polyhedron *P* is a formula where the planar half spaces corresponding to the faces of *P* appear as literals. In a *minimal Boolean* *formula* (MBF) or Peterson Style Formula [DG88], [Pe84], each planar half space appears exactly once. The planar half spaces completely describe *P* in the neighbourhood of their respective faces, and so a Boolean combination of them will describe *P* as a whole. If none of the half spaces coincide, none can be omitted.

Similarly, an MBF is defined for a vertex. For a spherical polygon, half spaces are replaced by the corresponding hemispheres.

3. Known Polyhedra without MBF

For every simple 2D polygon, there exists at least one MBF, which can be constructed in time O(*n* log *n*), *n* being the number of edges [DG88]. In 3D, on the contrary, there exist polyhedra without MBF. This has been shown by several examples, which impose some additional conditions.

A first condition is that the polyhedron be simplicial, i.e. that the faces of the polyhedron be triangles. This is important because otherwise it is possible to triangulate the faces, increasing the number of literals. Indeed, would it suffice to triangulate the faces of a polyhedron to produce an MBF, this would allow to attribute the impossibility of an MBF not to polyhedra as such, but to the complicated faces of these polyhedra. However, this is not possible, as was shown by Dobkin et al. [DG88].

The second condition that can be imposed on a polyhedron without MBF is that all the faces involved meet at a single vertex. This was done by the author and Kunii in [DK89]. It shows that not only 3D polyhedra, but also polygons on the sphere, which are inherently 2D objects, cannot be represented by an MBF. It thus more exactly draws the line between the set of objects with MBF and the set of objects with possibly no MBF. The drawback of this example was that it was not simplicial.

A polyhedron without MBF that does not adhere to any of the two above conditions is also possible [Sn91]. Its advantage is that it is very easy to see that it has no MBF.

4. A Simplicial Vertex without MBF

Until now, it remained open whether it is possible to impose both simplicity and incidence at a single vertex. This problem is answered positively by the example of Figure 1. This example polyhedron without MBF is bounded by five triangles (*A*...*E*) that meet at the vertex *V* and three triangles ($H_{1...3}$) on the convex hull that are used for completion.

A mapping of the corresponding spherical polygon to the plane is shown in Figure 2. That this indeed represents a simplicial vertex can be checked as follows: Every circle cuts all other circles twice, and both times in the same sequence. For five circles, all such arrangements are combinatorially equivalent. All boundary segments consist of less than four intervals between intersection points, which means that the corresponding angles at *V* are smaller than 180°.

For reasons of space, we have to omit the full proof that Figure 1 and 2 are indeed not representable by an MBF. The reader is referred to [Dü92]. Basically, the proof starts by considering all possible MBF, represented as expression trees, and successively excludes subsets of trees using various arguments, until no possible MBF is left.

It is interesting to note how this example vertex was discovered. First, a simplicial vertex with six faces was constructed by triangulating the example from [DK89]. Then the newly formed triangles were turned slightly towards each other so that they formed a reflex angle. The absence of an MBF for this configuration was verified mechanically by a Prolog program.

That there are no four-face simplicial vertices without MBF is easily verified, but there remained the possibility of a five-face vertex. A Prolog program again was used to check that Figure 2 is the only five-face non-MBF vertex, modulo rotations, mirroring, and complements.

That it is the smallest object without MBF can not only be shown for the five-face simplicial vertex, but also for the whole simplicial polyhedron of Figure 1 with eight faces. All simplicial polyhedra with four or six faces can be represented by an MBF.

5. Discussion

We have shown that there are simplicial vertices and spherical polygons that cannot be represented by a minimal Boolean formula (MBF). Thus not allowing an MBF for certain configurations is not only a property of 3D Euclidean space, but also of certain 2D spaces.

Several interesting questions remain open. The size of a Boolean formula for a spherical polyhedron is $\leq 2n$, as a sphere can be split into two hemispheres, for which the planar solution is applicable. However, is this optimal? Are there conditions that allow to predict the presence and absence of an MBF for a vertex? How and to what extend can Boolean formulæ for vertices, even if they are not MBF, be combined to solutions for general polyhedra?

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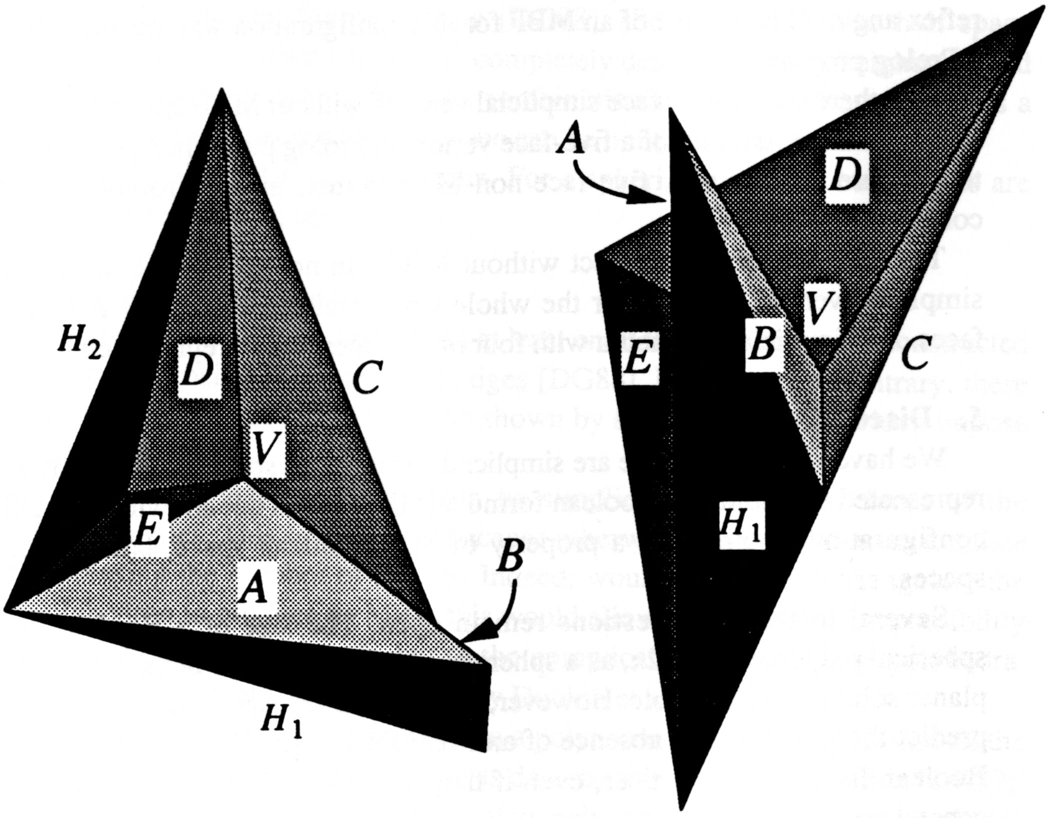

Figure 1. Two views of the polyhedron without MBF.

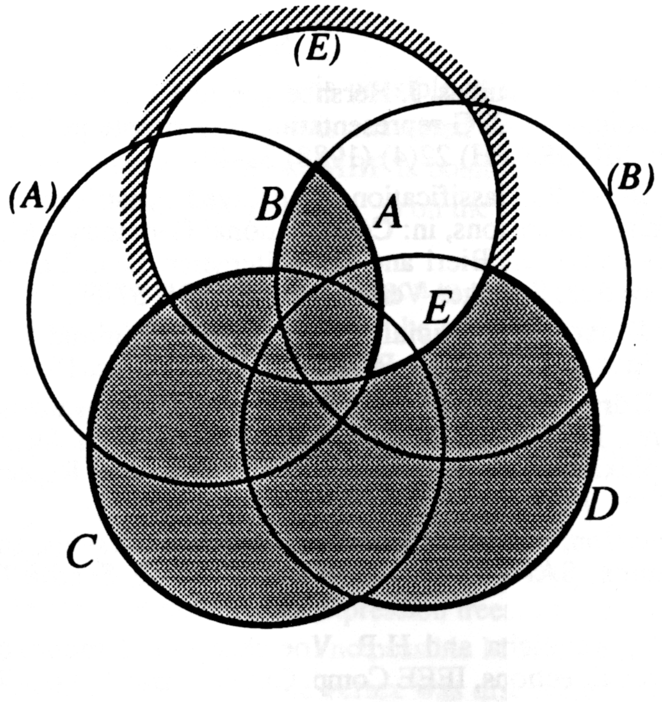

Figure 2. Venn-diagram of the vertex without MBF.
All literals, with the exception of *E*, denote the inside of the circles.