ON THE TOPOLOGICAL SHAPE OF PLANAR VORONOI DIAGRAMS *

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Abstract.- *Voronoi diagrams in the plane for strictly convex distances have been studied in* [3], [5] *and* [7]. *These distances induce the usual topology in the plane and, moreover, the Voronoi diagrams they produce enjoy of many of the good properties of Voronoi diagrams for the Euclidean distance. Nevertheless, we show (Th.1) that it is not possible to transform, by means of a bijection from the plane into itself, the computation of their Voronoi diagrams, to the computation of an Euclidean Voronoi diagram (except in the trivial case of the distance being affinely equivalent to the Euclidean distance). The same applies if we want to compute just the topological shape of a Voronoi diagram of at least four points (Th. 2).*

Moreover, for any strictly convex distance not affinely equivalent to the Euclidean distance, new, non Euclidean shapes appear for Voronoi diagrams, and we show a general construction of a nine-point Voronoi diagram with non Euclidean shape (Th.3).

Given a partition V of the plane into finitely many regions Ash and Bolker [1] have studied the problem of deciding if V is an Euclidean Voronoi diagram for some set of points (see also [2] and [5]). We can relax the conditions, and ask if the given partition V has at least the same topological shape of an Euclidean Voronoi diagram of some finite set of points. Here and in what follows we say that two cellular decompositions of the plane, each with a finite number of cells, have the *same topological shape* if there is an homeomorphism of the plane onto itself sending cells to cells.

This question is theoretically quite easy, as one can construct an algorithm to decide it as follows: taking the coordinates of the points for the Voronoi diagram as indeterminates, the fact that the Voronoi diagram for these points has the shape of V can be expressed as a finite set of conditions on these indeterminates, in such a way that there exists an Euclidean diagram with the shape of V if and only if the conditions are satisfied for some values of the indeterminates. Now, the conditions appearing are always about the position of the circle passing by three of the points respect to a fourth one, i.e. they are polynomial equalities or inequalities, and real quantifier elimination gives the answer to the problem whether they have a solution or not.

More interesting is to study the question about having the same topological shape of a planar Euclidean Voronoi diagram for the entire collection of partitions V arising as Voronoi diagrams for a non Euclidean distance. Does the changing of the distance imply a drastic change on the shape of Voronoi diagrams? Concretely we will consider the class of normed distances verifying the strong triangle inequality (i.e. the triangle equality holds only for collinear points, cf. [6]). Voronoi diagrams for these distances (that we shall call *strictly convex distances*) have been first considered by Chew and Drysdale [3] and then studied by Klein [5], Mazon [8], exhibiting an algorithm for their computation. Moreover, these distances are in many other respects quite close to the Euclidean distance; for instance they yield the usual topology on the plane and the Voronoi diagrams for them induce the same kind of cellular decomposition of the plane as the Euclidean Voronoi diagrams do. Thus the problem we posed about the conservation of the topological shape of Voronoi diagrams is quite natural in this situation.

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To study this problem we give some necessary conditions for a cellular decomposition to have the same topological shape of a Voronoi diagram for a strictly convex distance, and then exhaustively study the diagrams which satisfy them for low number of regions. We find that, even for four regions, there are some of them not "realizable" by any Euclidean Voronoi diagram, and that can be realized by some strictly convex distances. These four-points examples are based on the fact that if a strictly convex distance d is not *smooth* (in the sense that its unit ball has 'corners'), then three non collinear points need not to be d-cocircullar. If we restrict ourselves to smooth distances, up to six points the only topological shapes that the Voronoi diagrams can have are those of Euclidean Voronoi diagrams, but again some new shapes appear for seven points. Finally we prove that these kind of "non Euclidean shapes" for Voronoi diagrams appear for any strictly convex distance d, provided that d is not an affine transformation of the Euclidean distance. (A general construction of a nine point Voronoi diagram with different shape from any Euclidean Voronoi diagram is shown).

Another interesting problem is deciding, if two distances d and δ are given , whether it is possible or not to reduce by means of a bijection f in the plane, the computation of the Voronoi diagram of a set S of points for distance d to the computation of Voronoi diagram of the transformed points $f(S)$ for distance δ. At this respect our main results are:

Theorem 1. Let d and δ be two strictly convex distances in the plane.

(i) If $d = \delta \circ f$ with f a bijection of the plane onto itself (i.e. $d(P,Q) = \delta(f(P), f(Q)), \forall P, Q$), then f is an affine mapping, and for every finite set $S \subset \mathbb{R}^2$:

$$f(Vor_d(S)) = Vor_\delta(f(S)).$$

(ii) If there exists a bijection $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ preserving the bisectors of every two points, i.e. such that:

$$\forall P, Q \in \mathbb{R}^2 \qquad f(Bi_d(P,Q)) = Bi_\delta(f(P), f(Q)),$$

then f is affine, $d = k \cdot \delta \circ f = \delta \circ (k \cdot f)$, for some constant $k > 0$, and thus we are in the conditions of (i).

Two distances such that $d = \delta \circ f$, with f an affine bijection will be called *affinely equivalent.* Part (i) of Theorem 1 says that if one knows how to compute Voronoi diagrams for a given distance δ, then one can also compute them for any other affinely equivalent distance d. For instance, the problem of computing Voronoi diagrams with respect to a strictly convex distance d whose unit ball is an ellipse can be reduced to compute Euclidean Voronoi diagrams.

Part (ii) of Theorem 1 establishes that, in order that two given strictly convex distances d and δ are affinely equivalent, it suffices that a bijection f from the plane to the plane exists such that it preserves bisectors (which are two point Voronoi diagrams). In this case, (i) implies that the Voronoi diagram of any finite set of points will be also preserved. In other words, (ii) is a strong reciprocal of (i): the only transformations which allow to reduce the computation of the Voronoi diagram for one strictly convex distance to another one are bijective affinities. Note also that if we take the bijection f as being the identity, (i) and (ii) say that two distances produce identical Voronoi diagrams for every finite collection of points if and only if they have the same bisectors and that, in this case, they are related by $d = k\delta$ and so they have the same circles (a *circle* for a distance d is the set of points with equal distance to a fixed center; if it is be convinient to specify the distance we shall call them d-circles).

After this, in some sense, negative result, we are interested in knowing when this procedure of reduction permits to obtain, if not the exact diagrams, at least their topological shape, as this is the hardest part in the computation of a Voronoi diagram ([4]). We find the next negative result, with the additional hypothesis of the distances being smooth (i.e. with smooth circles).

Theorem 2. Let d and δ be two strictly convex and smooth distances in the plane. If there exists a bijection f from the plane onto itself such that for every finite set S, $Vor_d(S)$ has the same topological shape as $Vor_\delta(f(S))$, then f is affine, $d = k\delta \circ f$ for some constant $k > 0$ and we are in the conditions of Th.1(i). Moreover, the hypothesis is only needed for sets S of four or less points, and it is not sufficient to have it for sets of three points.

As a corollary to Theorems 1 and 2, to look for homeomorphism between Voronoi diagrams of strictly convex smooth distances is the same as to look for equality. We want to remark that the additional hypothesis of the distances being smooth is used in our proof, but possibly Theorem 2 would be still true without making it.

Finally we may wonder whether two strictly convex distances produce the same collection of topological shapes for Voronoi diagrams or not. We do not have a general answer for this, except if one of the distances is the Euclidean one, and in this case the answer is positive only for distances affinely equivalent to the Euclidean distance:

Theorem 3. If d is a strictly convex distance, not affinely equivalent to the Euclidean distance, then there exists some collection S of nine points whose Voronoi diagram with respect to distance d, $Vor_d(S)$, has not Euclidean shape.

2. References

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