Gray Code Enumeration of Plane Straight-Line Graphs*

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Abstract

We develop Gray code enumeration schemes for geometric graphs in the plane. The considered graph classes include plane straight-line graphs, plane spanning trees, and connected plane straight-line graphs. Previous results were restricted to the case where the underlying vertex set is in convex position.

1 Introduction

Let $E = \{e_1, \ldots, e_m\}$ be an ordered set. For the purposes of this paper, E will consist of the $m = \binom{n}{2}$ line segments spanned by a set S of n points in the plane, in lexicographical order. Consider a collection $\mathcal{A}$ of subsets of E. For instance, think of $\mathcal{A}$ being the class of all crossing-free spanning trees of S. We associate each member $A_i \in \mathcal{A}$ with its containment vector b_i with respect to E. That is, b_i is a binary string of length m whose j^{th} bit is 1 if $e_j \in A_i$ and 0, otherwise. A (combinatorial) *Gray code* for the class $\mathcal{A}$ is an ordering $A_1, \ldots, A_t$ of $\mathcal{A}$ such that b_{i+1} differs from b_i by a transposition, for $i = 1, \ldots, t-1$. For example (and as one of the results of this paper), for crossing-free spanning trees a Gray code exists such that successive trees differ by a single edge move. Depending on the class we will consider, a transposition will be an exchange of two different bits (as for the spanning tree class) or/and a change of a single bit. Combinatorial Gray codes generalize the classical binary reflected Gray code scheme [13] for listing m-bit binary numbers so that successive numbers differ in exactly one bit position. See [14] for a survey article. We also refer to [3] for various results concerning edge moves in spanning trees.

Any Gray code for a given class $\mathcal{A}$ provides a complete enumeration scheme for $\mathcal{A}$ by means of constant-size operations. Listing all the objects of a given class is a fundamental problem in combinatorics and, in particular, in computational geometry. Not every enumeration scheme constitutes a Gray code, however, as a small difference between consecutive objects will not be guaranteed, in general. For instance, the popular reverse search enumeration technique [6] lacks this property.

When interpreting $\{A_1, \ldots, A_t\}$ as the set of nodes of an abstract graph that connects two nodes A_i and A_j whenever b_i and b_j are a single transposition apart, any Gray code for the class $\mathcal{A}$ corresponds to a Hamiltonian path in this transposition graph. For example, if $\mathcal{A} = \mathcal{G}$, the class of all possible straight-line graphs on a point set S, then each of the 2^m subsets of E defines a member of $\mathcal{G}$, for $m = \binom{n}{2}$. The transposition graph for $\mathcal{G}$ is the hypercube in m dimensions.

While general enumeration schemes for geometric objects have been studied quite extensively, see e.g. [6, 2, 8, 7], respective results for Gray codes are sparse. In particular, all the known results for straight-line graphs concern the special case where the underlying set S of points is in convex position [5, 9, 11, 10]. These Gray code constructions are mainly based on a hierarchy of graphs structured by increasing point set cardinality. In this paper, we extend this approach to general point sets S, which becomes possible when combining it with classical combinatorial Gray codes. We construct Gray codes for the class $\mathcal{PG}$ of all plane straight-line graphs, the class $\mathcal{CPG}$ of all plane and connected straight-line graphs, and the class $\mathcal{ST}$ of all plane spanning trees, for a given point set. For the class $\mathcal{CPG}$ no results existed even for the convex case. The respective challenging question for triangulations remains open (for $n \geq 7$).

2 A hierarchy for plane graphs

Let $S = \{p_1, \ldots, p_n\}$ be the underlying set of points in the plane. Without loss of generality, let S be given in sorted order of x-coordinates. For simplicity, we also assume that no three points in S are collinear. Then, for $1 \leq k \leq n-1$, the point p_{k+1} lies outside the convex hull of $\{p_1, \ldots, p_k\}$. This property will turn out useful in the subsequent constructions.

Let now $\mathcal{A}$ be one of the classes $\mathcal{PG}$, $\mathcal{CPG}$, or $\mathcal{ST}$. We define a hierarchy (a tree structure) $H_{\mathcal{A}}(S)$ for $\mathcal{A}$ and S such that the k^{th} level of the hierarchy consists of all the members of $\mathcal{A}$ on top of the first k points in S, for $k = 1, \ldots, n$. That is, each member in $H_{\mathcal{A}}(S)$ except the root (at level 1) has a unique par-

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ent, and each member in $H_{\mathcal{A}}(S)$ which is not a leaf has a unique set of children.

The Gray code construction for $\mathcal{A}$ is done recursively. It hinges on an appropriate rule for defining the parent of a given member, as well as on a consistent rule for enumerating its children. Designing the enumeration rule for the children is the crucial part, as it has to yield a Gray code for the children which fits the (previously constructed) Gray code for the parents. In particular, $H_{\mathcal{A}}(S)$ has to be an ordered tree such that the ordering at each of its levels is a Gray code.

3 The class $\mathcal{PG}$

For the class $\mathcal{PG}$ of all plane straight-line graphs, things are surprisingly simple. Let G' be at level $k+1$ of the hierarchy $H_{\mathcal{PG}}(S)$, for $k \geq 1$. That is, G' is some plane straight-line graph whose vertex set is $\{p_1, \ldots, p_{k+1}\}$.

The parent of G' is obtained by removing from G' the vertex p_{k+1} and all its incident edges. This gives a unique graph G at level k of $H_{\mathcal{PG}}(S)$. We say that a vertex p_j of G is *visible* (from p_{k+1}) if the line segment $p_{k+1}p_j$ does not cross any edge of G. As we are interested only in plane graphs, all the children of G are obtained by adding the vertex p_{k+1} and connecting p_{k+1} to subsets of visible points in all possible ways (including the empty set).

Let v_G be the number of visible vertices of G. We can use the cyclic binary v_G-bit Gray code $B(v_G)$ (see Appendix) for encoding all the possible subsets of edges incident to p_{k+1}. A transposition then corresponds to adding or removing a single edge to a visible vertex. To specify the first and the last subset (i.e., child of G), we take $\emptyset$ for the first child and the singleton set $\{p_{k+1}p_j\}$ for the last child, where $p_{k+1}p_j$ is the edge tangent to the convex hull of G from above. (Vertex p_j is visible for any choice of the graph G. In particular, p_j is the first visible vertex in counterclockwise order.) Accordingly, the first string in the code $B(v_G)$ is $00\ldots0$ and the last string is $10\ldots0$.

To construct a Gray code for level $k+1$ of $H_{\mathcal{PG}}(S)$, let G and F be adjacent at level k of $H_{\mathcal{PG}}(S)$. Attaching $00\ldots0$ to the string for G and F, respectively, leaves the strings one transposition apart. The same is true for $10\ldots0$. That is, the first child of G is adjacent to the first child of F, and the last child of G is adjacent to the last child of F. So we can run $B(v_G)$ followed by $\overline{B(v_F)}$ (the code $B(v_F)$ in reverse order), and so on. The construction of the desired Gray code is now obvious. In addition, we can use the fact that the number of plane straight-line graphs on $n \geq 2$ points is even (because each convex hull edge appears in exactly half of the graphs), which by induction implies that the Gray code is cyclic. In the language of graph theory, our result reads:

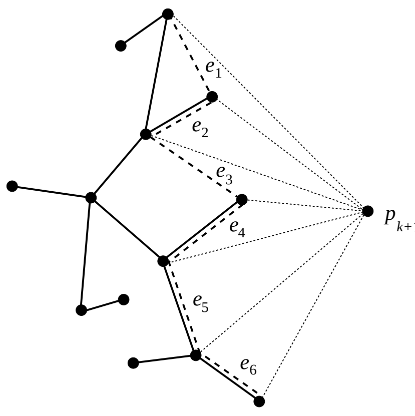

Figure 1: The chain of a spanning tree

Theorem 1 *The transposition graph for $\mathcal{PG}$ contains a Hamiltonian cycle.*

It would be interesting to know the average degree of p_{k+1} in a level-$(k+1)$ member of $H_{\mathcal{PG}}(S)$. This information could be used to give a lower bound on the number of plane straight-line graphs.

4 Plane spanning trees

Our strategy for constructing Gray codes also works for the class $\mathcal{ST}$ of plane spanning trees. A transposition will now be an exchange of two different bits, because the addition (or removal) of a single edge destroys the tree property.

Consider a member T' at level $k+1$ of the hierarchy $H_{\mathcal{ST}}(S)$, for $k \geq 1$. That is, T' is some plane spanning tree on $\{p_1, \ldots, p_{k+1}\}$. Defining an appropriate parent of T' is less trivial now.

For an arbitrary plane straight-line graph G on $\{p_1, \ldots, p_k\}$, let $q_1, \ldots, q_v$ be the visible vertices of G (as seen from p_{k+1}) in counter-clockwise order. We define the *chain for* G, $C(G)$, as the ordered set of line segments q_iq_{i+1}, for $1 \leq i \leq v-1$. Observe that for every edge $e \in C(G) \setminus G$, the set of visible vertices of $G \cup \{e\}$ is the same as for G.

Parent rule: Let G be the graph obtained by removing from T' the vertex p_{k+1} and all its incident edges. Let G consist of $r \geq 1$ components. We add the first $r-1$ edges of the chain $C(G)$ that connect these components, and define the resulting tree as the parent of T'.

This rule yields a unique parent for T'. Notice that this parent is well-defined (i.e., belongs to level k of $H_{\mathcal{ST}}(S)$) because G can always be connected to a tree using edges of $C(G)$, and no such edge crosses any edge of G. From the definition of the parent we get the definition for the children, as follows.

Let T be a tree at level k of the hierarchy $H_{\mathcal{ST}}(S)$, and let '$<$' denote the total order on $C(T)$. Define $E(T)$ as the set of all edges $e \in T \cap C(T)$ which satisfy the following two conditions:

(1) Removal of e does not make a non-visible vertex of T visible.

(2) No edge $e' \in C(T) \setminus T$ with $e' < e$ gives a cycle in $T \cup \{e'\}$ that contains e.

See Figure 1. The spanning tree T is drawn with bold lines. Its chain $C(T)$ consists of six edges $e_1, \ldots, e_6$. Edges e_1 and e_3 are not part of T and thus do not belong to $E(T)$. Also, we have $e_4, e_5 \notin E(T)$ because these edges reveal visible points. Finally, $e_2 \notin E(T)$ as the edge $e_1 < e_2$ closes a cycle in T that contains e_2. This gives $E(T) = \{e_6\}$.

Children rule: The children of T are obtained by removing, for each possible subset of $E(T)$, its edges from T and connecting each resulting component to the vertex p_{k+1} using a single edge to some visible vertex, in all possible ways.

It is clear that all the graphs constructed from T in this way are plane spanning trees on $\{p_1, \ldots, p_{k+1}\}$. To see that each member of level $k+1$ of the hierarchy $H_{\mathcal{ST}}(S)$ is generated exactly once by the children rule (provided all the members of level k have so), we need the lemma below. Let us color the edges of $E(T)$ *green*, and the edges that connect to p_{k+1} *red*.

Lemma 2 *The parent rule and the children rule are consistent.*

Proof. Child → parent: Let T' be some child constructed from T. Then T' contains $r \geq 1$ red edges. If $r = 1$ then no green edge has been removed from T for constructing T'. According to the parent rule, we now remove from T' the vertex p_{k+1}, which correctly gives us the single component T. Now assume $r \geq 2$. Then $r-1$ green edges have been removed from T to construct T', leaving r components. Let K_i and K_{i+1} be two of these components, such that there is a single (removed) green edge e between them. There is no edge $e' \in C(T) \setminus T$ with $e' < e$ and which gives a cycle in $T \cup \{e'\}$ containing e. Thus, when removing from T' the vertex p_{k+1} and joining components of the resulting graph G according to the parent rule, the edge e is indeed the first edge of $C(G)$ that connects K_i and K_{i+1}. (Otherwise we would have chosen e' for removal instead of e.) Thus we correctly get T from T' again.

Parent → child: Let T be the parent constructed from T'. T is obtained by removing p_{k+1} and joining the r components of the resulting graph G with edges of $C(G)$. Let e be such an edge. Then e is the first edge of $C(G)$ that connects the respective two components. Therefore, no edge $e' \in C(T) \setminus T$ with $e' < e$ gives a cycle in $T \cup \{e'\}$ that contains e. Moreover, because $e \in C(G) \setminus G$, we know that $G \cup \{e\}$ and G have the same set of visible vertices. In conclusion, e is indeed a green edge. Thus, by the children rule, T' will be constructed as one of the children of T. □

Theorem 3 *The transposition graph for $\mathcal{ST}$ contains a Hamiltonian path.*

Proof. Let T be a tree at level k. Define as the first child T^f (respectively, as the last child T^ℓ) of T the tree obtained when adding to T the upper (respectively, lower) tangent from p_{k+1} to the convex hull of T. Note that T^f and T^ℓ are well-defined children of T. We claim (and prove below) that the transposition graph for $\mathcal{ST}$ contains a path from T^f to T^ℓ that contains all the children of T. The theorem follows because, for two neighboring trees T and U at level k, their first children T^f and U^f, as well as their last children T^ℓ and U^ℓ, are a single transposition apart.

To encode all the children of T, we have to consider each subset $Y \subset E(T)$ of green edges and, for fixed Y, all allowed distributions of red edges. Let $g = |E(T)|$. Similar to Section 3, the binary reflected g-bit Gray code $B(g)$ (let us call it the *green* code) is used for encoding all possible subsets of $E(T)$. Now consider a fixed subset Y. Let $T \setminus Y$ have $r \geq 1$ components. Given an arbitrary visible vertex q_j in each component K_j, $1 \leq j \leq r$, there exists a Gray code $R(r)$ (the *red* code) that starts with the positions of the edges $q_1p_{k+1}, \ldots, q_rp_{k+1}$ and that encodes all allowed positions of the red edges; see the Appendix.

Between every two transpositions in the green code we work off the red code. Care has to be taken when switching between the codes. If a green edge is added (i.e., two components are joined) then some red edge has to be removed. All other red edges stay at their positions, which are the starting positions for the subsequent red code. Similarly, if a green edge is removed (i.e., a component is split into two) then some red edge has to be added. Again, all other red edges stay at their positions which are the starting positions for the next red code.

The green code is cyclic, and thus we obtain a cyclic Gray code for the children of T when combining the green and the red code. If now T^f and T^ℓ are not adjacent in this code, this property can be enforced as follows: The green code is started at the subset $Y = \emptyset$ of $E(T)$ (recall that T^f and T^ℓ arise as children for this subset) and the red code for $\emptyset$ is started with some red edge which is not a convex hull tangent. Such an edge exists because T^f and T^ℓ are already adjacent, otherwise. We first do one transposition in the green code and, after having returned to $\emptyset$ with the combined code in the end, we do the red code for $\emptyset$. Cutting the obtained code, in which T^f and T^ℓ are adjacent now, between these two trees gives the desired result. □

If the Hamiltonian path in Theorem 3 both starts and ends with a first child (or with a last child), then we get a Hamiltonian cycle. Constructing such a cycle in the general case is left to further research.

The Gray code construction above can be modified to yield a Gray code for the class $\mathcal{CPG}$ of all connected plane straight-line graphs. Allowed transpositions then are an exchange of two different bits or a change of a single bit. Disallowing either type makes the transposition graph for $\mathcal{CPG}$ non-Hamiltonian. Details are given in a full version of this paper.

5 Algorithmic issues

To perform a Gray code enumeration of a given graph class $\mathcal{A}$, the hierarchy $H_{\mathcal{A}}(S)$ is traversed in preorder and its leaves are reported. For $\mathcal{A} \in \{\mathcal{PG}, \mathcal{ST}, \mathcal{CPG}\}$, each non-leaf member of $H_{\mathcal{A}}(S)$ has at least two children. Consequently, the time complexity is dominated by computing the leaves. When computing the children of a given parent G, the main tasks are calculating the chain $C(G)$ and the set $E(G)$ of green edges. Both tasks can be accomplished in $O(|S|)$ time, by traversing the faces that G defines within its convex hull. We omit the details due to lack of space.

Theorem 4 *Let $\mathcal{A}$ be any of the classes $\mathcal{PG}$, $\mathcal{ST}$, or $\mathcal{CPG}$, for a given set S of n points in the plane. A Gray code enumeration of $\mathcal{A}$ can be performed in $O(n)$ time per member.*

For the class $\mathcal{ST}$ this result is superior to the $O(n^3)$ result for (non-Gray code) enumeration in [6].

6 Discussion

For several classes of plane straight-line graphs their transposition graph fails to be Hamiltonian, in general. E.g., for the class of all triangulations of a point set S, with edge flips [12] as transpositions, there is a counterexample for $|S| = 6$. However, it still may be that Hamiltonicity is attained when S is sufficently large. For pseudo-triangulations, see e.g. [1], the situation is unclear as well. Our conjecture is that the flip graph is Hamiltonian when both edge-exchanging *and* edge-inserting flips are admitted. The flip graph of minimum (or pointed) pseudo-triangulations and edge-exchanging flips is known to be Hamiltonian for all sets of up to 5 points. Our hierarchical approach possibly may be used to settle these problems.

With small adaptions, our enumeration scheme also works for point sets containing collinear points. Thus we can provide Gray codes for the respective graph classes on the grid.

The efficient generation of random spanning trees, plane graphs, or connected plane graphs is still an open problem. Progress can possibly be made by selecting random children for members in the respective classes.

7 Appendix

Binary reflected Gray code. The binary reflected Gray code $B(n)$ for n-bit numbers is defined recursively as follows. $B(1)$ is a list of two one-bit strings, namely $0, 1$. For $n \geq 2$, $B(n)$ is formed by taking the list for $B(n-1)$, prepending the bit 0 to every string, and then taking the list for $B(n-1)$ in reversed order and prepending the bit 1 to every string. Thus

$$B(n) = \begin{cases} 0, 1 & \text{if } n = 1 \\ 0 \cdot B(n-1) \ \circ \ 1 \cdot \overline{B(n-1)} & \text{if } n \geq 2 \end{cases}$$

with $\cdot$ and $\circ$ denoting character and list concatenation, respectively.

Red Gray code. For a given set Y of green edges, let the graph $T \setminus Y$ consist of the components $K_1, \ldots, K_r$. Let component K_j have v_j visible vertices, and let $1, \ldots, v_j$ be any fixed ordering for the positions of these vertices. The red code $R(1)$ for the first component K_1 is given by $1, 2, \ldots, v_1$. Assume we are given a red code $R(s)$ for $K_1, \ldots, K_s$, for $s < r$. Then the red code $R(s+1)$ is given by

$$1 \cdot R(s) \ \circ \ 2 \cdot \overline{R(s)} \ \circ \ 3 \cdot R(s) \circ \cdots \circ v_{s+1} \cdot \overline{R(s)}$$

where the last sublist is $v_{s+1} \cdot R(s)$ if v_{s+1} is odd.

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